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Constructing the identities and the central identities of degree < 9 of the $n \times n$ matrices

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CHAPTER 1. INTRODUCTION

Let $M_n(\phi)$ be the ring of $n \times n$ matrices with entries from a field ϕ. In a paper published in 1950, Amitsur and Levitzki [5] showed that $S_{2n}(x_1, ..., x_n)$ $= \sum_{\pi \in S_{2n}} sgn(\pi).[x_1.x_2...x_{2n}]_\pi$ vanishes for any choice of $M_1, ..., M_{2n} \in M_n(\phi)$. This 'multilinear identity' is known as the standard identitiy of the $n \times n$ matrices.

On June 6–8, 1956 a conference on linear algebra was held in Long Island, New York by the Division of Mathematics of the National Academy of Sciences–National Research Conference. In this conference, Kaplansky [11] proposed twelve problems in the theory of rings. One of the mentioned problems was the following: "Let A_n denote the $n \times n$ total matrix algebra over a field. Does there exist a polynomial which always takes values in the center of A_n without being identically zero?" The problem was stated carelessly since the constant polynomial or the constant polynomial added to the standard polynomial provides an answer to the problem. Moreover, the well known polynomial $(xy - yx)^2$ solves the problem for $n = 2$.

To avoid the above mentioned trivial cases, Kaplansky [12] rephrased the problem in 1970 as follows: "Let A_n denote the $n \times n$ total matrix algebra over a field, $n \geq 3$. Does there exist a homogeneous multilinear polynomial of positive degree which always takes values in the center of A_n without being identically zero?" The same problem was also brought up in the 10–th All–Union Algebra Colloquium, which took place on September 20, 1969 in Novosibirsk, Russia. In July 1972, Formanek [7] showed that the answer to the Kaplansky's problem is positive by constructing a central identity of degree $\frac{n(n+1)}{2}$ for each algebra $M_n(\phi)$. In December of the same year, Razmyslov [16] found a 'finite generating set' for the identities of $M_2(\phi)$, i.e., the identities of the matrix algebra of the second order over a field of characteristic zero. In 1973, Razmyslov [17] also constructed a new central identity of degree $3n^2 - 1$.

However, despite all the above and other attempts, the problem of describing all the identities and central identities of $M_n(\phi)$ for $n \geq 3$ has remained an open problem for many years. In this paper, we will describe a powerful method which enables us to find an 'independent generating set' for all the identities and central identities of degree $m < 9$ of the algebra $M_3(\phi)$, where ϕ is a field of characteristic zero or $p > m$.

The method requires knowledge of the group representation theory and relies heavily on computational techniques. It is worth mentioning that the method was first used to find a set of independent multilinear identities of the $M_2(Q)$ algebra, where Q is the field of rationals, and then the result was checked by comparing the identities with Razmyslov's identities.

CHAPTER 2. PROCEDURE

Basic Definitions and Concepts

Let ϕ be an associative and commutative ring with identity element 1. A ϕ-algebra is a ring R with identity element 1_R such that:

$$(i) \quad (R,+) \text{ is a unitary } \phi\text{-module} \tag{2.1}$$

$$(ii) \quad \alpha(rs) = (\alpha r)s = r(\alpha s) \qquad \text{for all } \alpha \in \phi \text{ and } r,s \in R \tag{2.2}$$

Let $X = \{ x_\alpha \}$ be an arbitrary set of symbols. we shall refer to the elements of X as variables. Now, we use juxtaposition on the elements of X to build up all possible finite sequences of the elements of X. We add parentheses to the sequences to make them admissible for a binary multiplication. Two sequences are equal if they are the same in every way. For example, the sequences $(x_1x_2)x_3$, $x_1(x_2x_3)$, and $(x_1{}^2x_2)x_3$ are all distinct. Such admissible sequences may be referred to as nonassociative sequences or words. Let $W[X]$ be the set of all finite nonassociative words created from the set X and $\phi[X]$ be the set of all linear combinations of elements of $W[X]$ with coefficients from ϕ. The elements of $W[X]$ and ϕ[X] are respectively called the nonassociative monomials and the nonassociative polynomials from the set X.

The degree of $x_i \in X$ in a monomial f is the number of times x_i appears in

f. The degree of f is defined to be the sum of the degrees of all the variables that appear in f. If f is a monomial in n variables $x_1, ..., x_n$, and the degree of x_i in f is k_i, then we say that f has type $[k_1, ..., k_n]$. For example, the monomial $f(x_1, x_2, x_3, x_4) = (x_1(x_2^3 x_4))x_1$ has type [2,3,0,1]. The degree of a nonassociative polynomial is the maximum degree of the degrees of all the monomials that appear in the polynomial. For example, the degree of $f(x_1, x_2, x_3) = (x_1^2 x_2)x_3 + x_1^4 + x_3(x_1^3 x_2)$ is 5. A multilinear monomial is a monomial of the type $[k_1, ..., k_n]$, where $k_i \leq 1$ for i=1,...,n.

A polynomial is said to be homogeneous in x_i if x_i has the same degree in all of the monomials appearing in the polynomial. A polynomial is called homogeneous if it is homogeneous in each variable. A homogeneous polynomial in which every monomial is multilinear is referred to as a multilinear polynomial. One can always express a polynomial f in the form $f = f_1 + f_2 + ... + f_k$, where each f_i is homogeneous. Each f_i is called a homogeneous component of f. If $f = \sum \alpha_i u_i$ and $g = \sum \beta_i v_i$ are any two elements of $\phi[X]$, the sum of f and g is obtained by adding coefficients of identical terms, and their product is given by:

$$fg = \sum \alpha_i \beta_j (u_i)(v_j) \tag{2.3}$$

The ring $\phi[X]$ is called the free nonassociative algebra on the set X over ϕ. Recall that the center $C(R)$ of a nonassociative ring R is the set of all elements $c \in R$ satisfying the following conditions for all $a, b \in R$:

$$(i) \quad ca = ac \tag{2.4}$$

$$(ii) \quad (ca)b = c(ab) \tag{2.5}$$

$$(iii) \quad (ac)b = a(cb) \tag{2.6}$$

$$(iv) \quad (ab)c = a(bc) \tag{2.7}$$

Now suppose $f = f(x_1, ..., x_n)$ is a polynomial in n variables. f is said to be an identity of the ϕ-algebra R if $f(x_1, ..., x_n) = 0$ for any choice of $x_1, ..., x_n \in R$. We may also say that R satisfies the polynomial f. f is called a central identity of the ϕ-algebra R if f is not an identity, and $f(x_1, ..., x_n) \in C(R)$ for all $x_1, ..., x_n \in R$.

Let $A = \{f_1, ..., f_k\}$ be a set of polynomials of $\phi[X]$. The T–ideal generated by A is the ideal of $\phi[X]$ generated by $\{f_i(y_1, ..., y_{n_i}) \mid i = 1, ..., k$ and $y_j \in \phi[X]$ for $j = 1, ..., n_i\}$. The elements in the T–ideal generated by A are called the identities implied by A. An identity of a ϕ-algebra R is said to be minimal if it is not implied by a set of identities of R of lower degrees. Two identities f and g are said to be equivalent if f implies g and vice versa. Similarly, we can define minimality and equivalence for the central identities.

In this chapter, we will give a complete procedure which enables us to find all the multilinear identities and the multilinear central identities of degree $n \leq 8$ or perhaps higher of the ϕ-algebra $M_3(\phi)$, where ϕ is the field of rationals or Z_p; p is any prime greater than 7 and $M_3(Q)$ is the ring of all 3×3 matrices with entries from ϕ. In chapter 4, we will prove that every identity or central identity of the above algebra is implied by the set of multilinear identities or central identities computed by the procedure. Consequently, it is sufficient to find only the multilinear identities and central identities.

The concepts related to the group representation theory of the symmetric group of n objects S_n play a very important role in the theory behind our procedure. The detailed proofs of all of the theorems are given in the last chapter. But before we state the procedure, we need a few basic definitions and concepts from the group representation theory.

A frame of degree n consists of n boxes arranged in such a way that $m_1 \geq m_2 \geq \ldots \geq m_r$, where m_i denotes the length of the i-th row for $i = 1, \ldots, r$. Thus for a frame with r rows and n boxes, we have:

$$m_1 + \ldots + m_r \quad = \quad n \tag{2.8}$$

We shall refer to such a frame by $(m_1, \ldots, m_r)$. Suppose two frames F and F' are respectively given by $(m_1, \ldots, m_r)$ and $(m'_1, \ldots, m'_k)$. Then we say that $F > F'$ if the first nonzero $m_i - m'_i$ is positive. The frames of a given degree n are always listed in decreasing order. Below, we have listed all the frames $F_{first}, \ldots, F_{last}$ of degree 5:

F_1 :
```
[ ][ ][ ][ ][ ]
```

F_2 :
```
[ ][ ][ ][ ]
[ ]
```

F_3 :
```
[ ][ ][ ]
[ ][ ]
```

F_4 :
```
[ ][ ][ ]
[ ]
[ ]
```

F_5 :
```
[ ][ ]
[ ][ ]
[ ]
```

F_6 :

```
[  ][  ]
[  ]
[  ]
[  ]
```

F_7 :

```
[  ]
[  ]
[  ]
[  ]
[  ]
```

A tableau of degree n is created by putting the numbers 1 to n into the n boxes of a frame. A standard tableau is a tableau in which the numbers are increasing in every row from left to right and in every column from top to bottom. We only need to number the standard tableaus for the theory of group representation. We shall enumerate the standard tableaus in the 'systematic' or 'dictionary' order. The 'nonstandard' tableaus are not numbered. As an example, we have listed all the standard tableaus $T_1, ..., T_f$ of the (3,2) frame.

T_1 :

```
[ 1 ][ 2 ][ 3 ]
[ 4 ][ 5 ]
```

T_2 :

```
[ 1 ][ 2 ][ 4 ]
[ 3 ][ 5 ]
```

T_3 :

```
[ 1 ][ 2 ][ 5 ]
[ 3 ][ 4 ]
```

T_4 :

[1][3][4]
[2][5]

T_5 :

[1][3][5]
[2][4]

A permutation $\pi \in S_n$ applied to a tableau T is simply a renumbering of the boxes in T and is denoted by πT. For example, let T_5 be the same as in the above example. Then the tableau $(1\ 2\ 4)T$ is given by:

[2][3][5]
[4][1]

A permutation is said to be a horizontal permutation or operation for a tableau T if it interchanges only the numbers of each row amongst themselves. We may also say that such permutation is a 'p' for T. Similarly, we can define a vertical permutation 'q' for T. The set of all horizontal permutations of a tableau T forms a group which we shall refer to by $\overline{P}$. The same fact holds for the set of all vertical permutations of T which is referred to by $\overline{Q}$. Obviously, the only permutation which is both a 'p' and a 'q' for a given tableau is the identity permutation, i.e.,

$$\overline{P} \cap \overline{Q} = \{ I \} \tag{2.9}$$

One may observe that if a permutation r moves a number from the (ij)–th position (i.e., the intersection of the i–th row and the j–th column) of a tableau T to the (kl)–th position, then the permutation $\pi r \pi^{-1}$ moves whatever number is at the (ij)–th position of πT to the (kl)–th position. Thus we have the following result:

Theorem 2.1. If r is a horizontal (vertical) permutation for a tableau T, then the permutation $\pi r \pi^{-1}$ is a horizontal (vertical) permutation for the tableau πT.

For a tableau T construct:

$$P = \sum_{p} p \quad \text{and} \quad Q = \sum_{q} sgn(q).q \tag{2.10}$$

The sums are taken over all horizontal and vertical permutations of T respectively. Now we define:

$$\bar{e} = PQ = \sum_{pq} sgn(q).pq \tag{2.11}$$

The sum is taken over all permutations of the form 'pq'. For instance, for the tableau

T_2 = [1][3]
[2][4] , we have:

$$\overline{e_2} = \{\, I + (1\ 3) + (2\ 4) + (1\ 3)(2\ 4)\,\}\,\{\, I - (1\ 2) - (3\ 4) + (1\ 2)(3\ 4)\,\}$$

One should also note that a permutation can be expressed at most in one way in the form 'pq' since $p_1 q_1 = p_2 q_2$ implies that $p_2^{-1} p_1 = q_2 q_1^{-1}$, and it follows from (2.9) that $p_1 = p_2$ and $q_1 = q_2$.

Theorem 2.2. Let T be a tableau with corresponding groups $\overline{P}$ and $\overline{Q}$. Then (1) : $pP = Pp = P$ and (2): $Qq = qQ = sgn(q)Q$.

Proof. (1): For any $p \in \overline{P}$, the map $f : \overline{P} \to \overline{P}$ given by $f(p') = pp'$ is one to one and onto. This implies that $pP = P$. Similarly, we have $Pp = P$.
(2): Let $q \in \overline{Q}$. As in above, we can show that qQ and Qq reproduce all the vertical permutations. Then

$$\begin{aligned} qQ &= q \sum_{q' \in \overline{Q}} sgn(q')q' \\ &= \sum_{q' \in \overline{Q}} sgn(q')qq' \\ &= \frac{1}{sgn(q)} \sum_{q \in \overline{Q}} sgn(qq')qq' \\ &= sgn(q)Q \end{aligned}$$

Similarly, we have $Qq = sgn(q)Q$.

$\square$

We can combine Theorems 2.1 and 2.2 to prove the following result.

Corollary 2.3. Let T be a tableau of degree n and $\pi \in S_n$. Then $P' = \pi P \pi^{-1}$, $Q' = \pi Q \pi^{-1}$ and $\overline{e'} = \pi \overline{e} \pi^{-1}$, where P', Q', and $\overline{e'}$ correspond to the tableau $T' = \pi T$.

Remark 2.4. pqT means first apply a vertical permutation q to the tableau T and then apply p to qT. Obviously, p may not correspond to a horizontal operation for the tableau qT. It is, however, correct to say that pqT is obtained from

T by first applying a horizontal permutation and then applying a vertical permutation. By Theorem 2.1, pqp^{-1} is a vertical permutation for the tableau pT. Thus the tableau pqT may be obtained by first applying a horizontal permutation p to T and then applying the vertical permutation pqp^{-1} to pT, i.e., $pqT = (pqp^{-1})pT$.

Let F be a fixed frame of degree n and T_i and T_j be any two standard tableaus belonging to F. The permutation that takes T_j to T_i is denoted by S_{ij} or $S_{i,j}$. In other words:

$$S_{ij}T_j = T_i \tag{2.12}$$

The inverse of S_{ij} is denoted by $S_{ij}{}^{-1}$, and we obviously have:

$$S_{ij}{}^{-1} = S_{ji} \tag{2.13}$$

Corollary 2.3 gives us the following useful formula:

$$\overline{e_i} = S_{ij}\overline{e_j}S_{ji} \tag{2.14}$$

Multiplying both sides of the above on the right by S_{ij}, we get:

$$\overline{e_i}S_{ij} = S_{ij}\overline{e_j} \tag{2.15}$$

Of course, the above relations remain valid if we replace $\overline{e_i}$ and $\overline{e_j}$ by $c\overline{e_i}$ and $c\overline{e_j}$ respectively, where c is any nonzero constant. For reasons which will be mentioned in future we take $c = \frac{d}{n!}$, where d is the dimension of the left ideal generated by e, and n is the degree of the frame. Now, for a tableau T_i define:

$$e_i = \frac{d}{n!}\overline{e_i} \tag{2.16}$$

Then equations (2.14) and (2.15) can be rewritten as follows:

$$e_i = S_{ij} e_j S_{ji} \tag{2.17}$$

$$e_i S_{ij} = S_{ij} e_j \tag{2.18}$$

Sometimes we may use a superscript to differentiate between the $e_i S_{ij}$'s belonging to the standard tableaus of different frames. For example, $e_i{}^k S_{ij}{}^k$ belongs to the k-th frame. If we are working with the standard tableaus of a certain frame, then for simplicity the superscript may be omitted from the notation.

In chapter 5, we will show that all the $e_i S_{ij}$'s of the standard tableaus of all the frames of a given degree n are linearly independent, and therefore, it is reasonable to express the multilinear identities and the central identities in terms of these elements.

General Approach

Our ultimate goal is to give a procedure that will enable us to find all the multilinear identities and the multilinear central identities of degree $n \leq 9$ of the Q-algebra $M_3(Q)$. The field of rationals Q may be replaced by any field of characteristic $p > 7$. In this chapter, we will use 'ϕ' to refer to Q or Z_p.

As we mentioned in the previous section, and we shall prove in chapter 4, it is sufficient to find only the multilinear identities and the multilinear central identities. Also, since the matrix rings are associative, we can drop the parentheses in the construction of the polynomials of the identities. A multilinear identity or a multilinear central identity of degree n is a polynomial of the form:

$$\sum_{\pi \in S_n} \alpha_\pi [x_1.x_2 \ldots x_n]_\pi \qquad \text{for some } \alpha_\pi\text{'s} \in \phi \tag{2.19}$$

or equivalently:

$$\sum_{\pi \in S_n} \alpha_\pi [x_{\pi^{-1}(1)} . x_{\pi^{-1}(2)} \ldots x_{\pi^{-1}(n)}] \tag{2.20}$$

The above multilinear polynomial identity can be represented by the following:

$$\sum_{\pi \in S_n} \alpha_\pi \pi \tag{2.21}$$

If $I(x_1, \ldots, x_n)$ is an identity of degree n, then so is $I\pi(x_1, \ldots, x_n) = I(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ for any permutation $\pi \in S_n$. If g is a linear combination of permutations, i.e., it is of the form $g = \sum_{\pi \in S_n} \alpha_\pi \pi$ for some α_π's $\in \phi$, then $Ig(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\pi I(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ is also an identity. We say that a set of multilinear identities $I_1(x_1, \ldots, x_n), \ldots, I_k(x_1, \ldots, x_n)$ is independent under substitution if for any choice of $g_1, \ldots, g_k$ of linear combinations of permutations of S_n, $I_1 g_1(x_1, \ldots, x_n) + \ldots + I_k g_k(x_1, \ldots, x_n) = 0$ would imply that $I_1 g_1(x_1, \ldots, x_n) = \ldots = I_k g_k(x_1, \ldots, x_n) = 0$. A set of multilinear identities which is independent under substitution and generates all the multilinear identities of degree n is called an independent generating set for all the multilinear identities of degree n. Similarly, we can define an independent generating set for all the multilinear central identities of degree n.

When we speak of the identities of a given frame, we mean all the multilinear identities of the form (2.21) which can be written as a linear combination of the $e_i S_{ij}$'s of all the standard tableaus of that frame. In the last chapter, we will prove that all the multilinear identities of all of the frames of a given degree n together can generate all the multilinear identities of degree n. Furthermore, we will show that identities computed by the procedure are independent under substitution, and they do indeed form an independent generating set for all of the identities of degree n. We will also show that the same argument holds for the multilinear central identities. In this section, we will discuss a general method of finding all the multilinear identities of a fixed frame. The method is analyzed in more detail in the next section.

Let F be a fixed frame of degree n with f standard tableaus $T_1, ..., T_f$. Define:

$$E_k \;=\; e_k S_{k1} \qquad k = 1, ..., f \tag{2.22}$$

In chapter 5, we will show that every multilinear identity or central identity of a frame F may be expressed as a linear combination of the E_k's of that frame. If $g = \sum_{\pi \in S_n} \alpha_\pi \pi$, then by $[x_1 ... x_n]_g$, we mean $\sum_{\pi \in S_n} \alpha_\pi [x_1.x_2...x_n]_\pi$. Then a multilinear identity of the given frame is of the following form:

$$I(x_1, ..., x_n) \;=\; \sum_{k=1}^{f} \alpha_k [x_1 ... x_n]_{E_k} \tag{2.23}$$

which is satsfied by any choice of n 3×3 matrices $M_1, ..., M_n$. We may also refer to an identity of the form (2.23) by $\sum_{k=1}^{f} \alpha_k E_k$. Now let:

$$h_k(M_1, ..., M_n) \;=\; [M_1 ... M_n]_{E_k} \tag{2.24}$$

Thus we want to find coefficients $\alpha_1, ..., \alpha_n \in \phi$ not all zero such that the following relation holds for any choice of n random matrices $M_1, ..., M_n \in M_3(\phi)$.

$$\sum_{k=1}^{f} \alpha_k h_k(M_1, ..., M_n) \;=\; 0 \tag{2.25}$$

Our process requires several choices of sets of 3×3 matrices, $\{M_1, ..., M_n\}$, $\{M_1', ..., M_n'\}$, ... For brevity, we will refer to $h_k(M_1, ..., M_n)$ by h_{ik}, where i is an integer that refers to the i–th choice set of n random matrices. Thus h_{ik}

is the evaluation of $[x_1...x_n]_{E_k}$ for a fixed set of n 3×3 matrices, and h_{lk} is the evaluation of the same polynomial for a different set of n 3×3 matrices.

We note that (2.25) gives us a system of nine equations; that is, one equation for each entry of the 3×3 matrix. Because of this, it is reasonable to consider the 3×3 matrices as 9×1 column vectors. In general, we let

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cong \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{31} \\ m_{32} \\ m_{33} \end{bmatrix} \tag{2.26}$$

Returning to (2.25), our problem simplifies to finding the solution $[\alpha_1, ..., \alpha_f]^T$ of the following system, where $[\ .\ .\ .\ ]^T$ denotes the transpose of $[\ .\ .\ .\ ]$.

$$\begin{bmatrix} h_{11} & \cdots & h_{1f} \\ \vdots & \ddots & \vdots \\ h_{r1} & \cdots & h_{rf} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.27}$$

Solving the above system of equations is equivalent to finding the nullspace of the following matrix:

$$\begin{bmatrix} h_{11} & \cdots & h_{1f} \\ \vdots & \ddots & \vdots \\ h_{r1} & \cdots & h_{rf} \end{bmatrix} \tag{2.28}$$

Each row of the matrix of (2.28) is evaluated for a set of n 3×3 random matrices, and therefore, it represents nine equations. As a result, for all practical purposes, it is regarded as a $(9r) \times f$ matrix. The number r is the number of trials, and it should be chosen large enough to make sure that the defined matrix has reached its maximum rank.

Procedure to Find All the Multilinear Identities of Degree N

As one may expect, the procedure involves extremely tedious and time consuming computations. Therefore, the only practical way of handling the problem is by computer. The procedure is originally designed to find an independent generating set for all of the multilinear identities of the Q-algebra $M_3(Q)$. Unfortunately, the numbers get larger as the degree of the identities gets larger. In general, one may encounter roundoff error or some type of numerical problem when the degree n becomes larger than 7. However, since the procedure is valid for any field Z_p where p is a prime greater than 7, one is advised to first excecute the procedure for several primes and find all the identities of their corresponding fields. Once all the multilinear identities of several of the Z_p-algebras of the ring $M_3(Z_p)$ are known, one can compare them to find all the multilinear identities of the Q-algebra of $M_3(Q)$. Finally, it should be mentioned that some of the steps given in the procedure may be combined together, and some others may be divided into several substeps. Here is the procedure.

Step 1

In this step, we create the first frame of degree n. Recall that the frames are always listed in the decreasing order. Therefore, the first frame F_1 is the 'horizontal' frame or the frame [ ][ ]...[ ]. This step initiates the procedure and is excecuted only once.

Step 2

Given a frame of degree n, we create the next frame of that degree. In general, one has to execute steps 3 to 9 to find all independent multilinear the identities of a fixed frame. Then one needs to create the next frame and find all the independent multilinear identities of the new frame and so on.

Step 3

We create all the standard tableaus of a given frame. In order to keep track of all the standard tableaus, one should create them in the dictionary order and then store them in $T_1, ..., T_f$.

The next step is the most complicated of all the steps, and therefore, it should be treated with tremendous care. The following lemma is used to justify the calculations involved in the next step of the procedure.

Lemma 2.5. $\sum_{p_k q_k} sgn(q_k) p_k q_k S_{k1} = \sum_{\pi \in S_n} \alpha_\pi \pi$, where $\alpha_\pi = sgn(q_k)$ if πT_1 can be obtained from T_k by applying a row operation followed by a column operation, and α_π is zero otherwise. p_k and q_k respectively denote horizontal and vertical permutations for T_k.

Proof. Given $\pi \in S_n$, we have:

$$\begin{aligned} & \pi \text{ is of the form } p_k q_k S_{k1} \\ \leftrightarrow \quad & p_k q_k S_{k1} T_1 = \pi T_1 \\ \leftrightarrow \quad & p_k q_k T_k = \pi T_1 \\ \leftrightarrow \quad & p_k q_k {p_k}^{-1} . p_k T_k = \pi T_1 \end{aligned}$$

Thus π is of the form $p_k q_k S_{k1}$ if and only if πT_1 can be obtained from T_k by a row followed by a column operation. If πT_1 can not be obtained from πT_k by a row followed by a column operation, then π is not of the form $p_k q_k S_{k1}$; so $\alpha_\pi = 0$. Otherwise $\alpha_\pi = sgn(q_k) = sgn(\tilde{q_k})$, where $(\tilde{q_k})$ is the unique column permutation for πT_1 which moves numbers to the same row they occupy in T_k.

□

<u>Step 4</u>

For a frame with f standard tableaus, we create $E_k = e_k S_{k1}$ for each $k = 1, ..., f$. Recall that $e_k = \sum_{p_k q_k} sgn(q_k) . p_k q_k$. Thus for each $k = 1, ..., f$, we have:

$$E_k = \sum_{p_k q_k} sgn(q_k).p_k q_k.S_{k1} \tag{2.29}$$

The sum is taken over all permutations $p_k q_k$ of the tableau T_k. According to Lemma 2.5, E_k can be expressed as $\sum_{\pi \in S_n} \alpha_\pi \pi$, where α_π is either zero or $sgn(\tilde{q_k})$ as given in Lemma 2.5. A reasonable outline to compute E_k is given below. In order to carry out this outline, one needs to write a program which creates all the permutations of S_n. More precisely, given a permutation of S_n, the program should create the 'next permutation' of S_n. The permutations should be created in some logical order to avoid any repetition. One also needs a program that compares any two tableaus T_i and T_j of a given frame. This program should determine whether or not T_j can be obtained from T_i by applying a row followed by a column operation, and if so, then it should determine the sgn of the column operation or $sgn(\tilde{q_k})$.

- (4 - i) Create the tableau πT_1 for a given $\pi \in S_n$.
- (4 - ii) Determine if πT_1 is obtained from T_k by a row operation p followed by a column operation $\tilde{q_k}$. If so, determine $sgn(\tilde{q_k})$ and let $\alpha_\pi = sgn(\tilde{q_k})$; otherwise let $\alpha_\pi = 0$.
- (4 - iii) Store the result in $sgn(\tilde{q_k})\pi$.
- (4 - iv) Repeat steps (4 - i) to (4 - iii) for each $\pi \in S_n$.
- (4 - v) Repeat steps (4 - i) to (4 - iv) to find E_k for each $k = 1, ..., f$.

Step 5

In this step, we create n 3×3 random matrices. As mentioned before, it is preferred to express each matrix as a column vector.

Step 6

Evaluate $E_1, ..., E_f$ for the random matrices created in step 5. This is the same as evaluating $h_{i1}, ..., h_{if}$, and therefore, it initiates the matrix given by (2.28). In order to achieve this, one obviously needs a program which multiplies the n random matrices in any order, i.e., a program which evaluates $[M_{\pi^{-1}(1)} M_{\pi^{-1}(2)} ... M_{\pi^{-1}(n)}]$ for any $\pi \in S_n$.

Step 7

Find the row canonical form of the matrix of (2.28). The objective, of course, is to find the nullspace of the mentioned matrix for a reasonable number of trials. The first time that the procedure is excecuted for a fixed frame, the matrix whose row canonical form is to be computed will be the same as the one that is initiated in step 6. The row canonical form is stored in some matrix. After that, every time that the procedure is excecuted, the new vectors $h_{i1}, ..., h_{if}$ of step 6 are placed at the bottom of the stored matrix, and the row canonical form of the resulting matrix is computed.

Step 8

We return to step 5. Steps 5, 6 and 7 are repeated until we are convinced that the row canonical matrix of step 7 has reached its maximum rank.

Step 9

We find the nullspace of the final matrix. This will give us all the possible set of coefficients $\alpha_1, ..., \alpha_f$ of $E_1, ..., E_f$, and therefore a set of linearly independent identities for the given frame. This is the only step which may not require intensive computer calculations since the matrix obtained from step 8 is already in the row canonical form.

Step 10

Now that a set of linearly independent multilinear identities of a given frame is known, we can go back to the second step and repeat the procedure to find a set of linearly independent identities of the next frame. This process is continued frame by frame until we find all the multilinear identities.

Procedure to Find All the Multilinear Central Identities of Degree N

By definition, $I(x_1, ..., x_n) = \sum_{k=1}^{f} [x_1 ... x_n]_{E_k}$ is a central identity of a given frame if $I(x_1, ..., x_n) \in C(M_3(\phi))$ for all $x_1, ..., x_n \in M_3(\phi)$ and is nonzero for some choice of $x_1, ..., x_n \in M_3(\phi)$. Also it is well known that the center of a matrix ring consists of all scalar multiples of the identity matrix. Therefore, $I(x_1, ..., x_n)$ is a central identity of the ϕ-algebra $M_3(\phi)$ if it is not an identity and for any choice of $M_1, .., M_n \in M_3(\phi)$, there exists a constant $c \in \phi$ such that:

$$I(M_1, ..., M_n) = c.I_{3\times 3} \tag{2.30}$$

which is the same as the following according to the notation (2.26).

$$I(M_1, ..., M_n) = (c \; 0 \; 0 \; 0 \; c \; 0 \; 0 \; 0 \; c)^T \tag{2.31}$$

One should also keep in mind that a central identity may be expressed as a linear combination of other identities and central identities. Let $h_k(M_1, ..., M_n)$ and h_{ik} for a frame F be defined as before. Then a multilinear central identity of the given frame must satisfy:

$$\sum_{k=1}^{f} \alpha_k h_{ik}(M_1, ..., M_n) \in C(M_3(\phi)) \tag{2.32}$$

for any choice of random matrices $M_1, ..., M_n \in M_3(\phi)$. Furthermore, the left hand side of (2.32) must be nonzero for some choice of n random 3×3 matrices. Now, let $h_{ik}(l)$ denote the l-th entry of the column vector $h_k(M_1, ..., M_n)$, i.e.,

$$h_{ik} = \begin{bmatrix} h_{ik}(1) \\ \vdots \\ h_{ik}(9) \end{bmatrix} \qquad k = 1, ..., f \tag{2.33}$$

If $I(x_1, ..., x_n)$ is a multilinear central identity, then the following relation must be satisfied for some $c \in \phi$:

$$\begin{bmatrix} h_{i1}(1) & \cdots & h_{if}(1) \\ \vdots & \ddots & \vdots \\ h_{i1}(9) & \cdots & h_{if}(9) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = c.I_{3\times3} \tag{2.34}$$

or equivalently:

$$\begin{bmatrix} h_{i1}(1) & \cdots & h_{if}(1) \\ h_{i1}(2) & \cdots & h_{if}(2) \\ h_{i1}(3) & \cdots & h_{if}(3) \\ h_{i1}(4) & \cdots & h_{if}(4) \\ h_{i1}(5)-h_{i1}(1) & \cdots & h_{if}(5)-h_{if}(1) \\ h_{i1}(6) & \cdots & h_{if}(6) \\ h_{i1}(7) & \cdots & h_{if}(7) \\ h_{i1}(8) & \cdots & h_{if}(8) \\ h_{i1}(9)-h_{i1}(1) & \cdots & h_{if}(9)-h_{if}(1) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.35}$$

Obviously, any multilinear identity or central identity satisfies the last 8 equations of the above system. In fact, for $c = 0$, the above system is equivalent to the 9 equations of the first row of the system given by (2.27). Define

$$\overline{g_{ik}} = \begin{bmatrix} h_{ik}(1) \\ h_{ik}(2) \\ h_{ik}(3) \\ h_{ik}(4) \\ h_{ik}(5)-h_{ik}(1) \\ h_{ik}(6) \\ h_{ik}(7) \\ h_{ik}(8) \\ h_{ik}(9)-h_{ik}(1) \end{bmatrix} \tag{2.36}$$

Now, consider the following system of equations:

$$\begin{bmatrix} \overline{g_{11}} & \cdots & \overline{g_{1f}} \\ \vdots & \ddots & \vdots \\ \overline{g_{r1}} & \cdots & \overline{g_{rf}} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.37}$$

where, as before, r is the number of trials. This system is obviously equivalent to the system given by (2.27). Therefore, the nullspace of the left hand side matrix in the above equation gives us all of the multilinear identities of the given frame. Let d_1 be the nullity of the matrix $[\overline{g_{ij}}]$, i.e., the number of the identities and

$$g_{ik} = \begin{bmatrix} 0 \\ \overline{g_{ik}}(2) \\ \overline{g_{ik}}(3) \\ \overline{g_{ik}}(4) \\ \overline{g_{ik}}(5) \\ \overline{g_{ik}}(6) \\ \overline{g_{ik}}(7) \\ \overline{g_{ik}}(8) \\ \overline{g_{ik}}(9) \end{bmatrix} = \begin{bmatrix} 0 \\ h_{ik}(2) \\ h_{ik}(3) \\ h_{ik}(4) \\ h_{ik}(5) - h_{ik}(1) \\ h_{ik}(6) \\ h_{ik}(7) \\ h_{ik}(8) \\ h_{ik}(9) - h_{ik}(1) \end{bmatrix} \tag{2.38}$$

From the previous remarks we know that if $[\alpha_1 \ldots \alpha_f]^T$ is a solution of the following system, then it could represent the coefficients of either an identity or a central identity. Of course, once the coefficients are known, one can easily determine whether the polynomial is an identity or a central identity by simply evaluating the polynomial for a few different sets of n random 3×3 matrices.

$$\begin{bmatrix} g_{11} & \cdots & g_{1f} \\ \vdots & \ddots & \vdots \\ g_{r1} & \cdots & g_{rf} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.39}$$

If $\alpha_1, ..., \alpha_f$ are coefficients of a multilinear central identity of frame F, then they must satisfy the above system. The same same is true for any multilinear identity of frame F, i.e., any solution of (2.37) is also a solution of (2.39). Thus if d_2 is the nullspace of the matrix $[g_{ij}]$, then we may say that there are exactly $d_1 - d_2$ many 'new' multilinear central identities of frame F. That means all the other multilinear central identities of the given frame can be obtained from the independent multilinear identities and the new multilinear central identities. For this reason, we may refer to the set of all independent multilinear identities and the new multilinear central identities of a given frame as a generating set for all of the multilinear central identities of that frame. Of course, the 'new' multilinear central identities are not unique, but one has to choose exactly $d_2 - d_1$ many of them, and make sure that they are linearly independent. Now, let us summarize the procedure to find an independent generating set for all of the multilinear central identities of a given frame.

<u>Step 1</u>

Use the procedure given in the previous section to find an independent generating set for all the multilinear identities of a given frame of degree n.

Step 2

In a separate file or directory, modify the previous procedure to find the nullspace of the following matrix for each frame.

$$\begin{bmatrix} g_{11} & \cdots & g_{1f} \\ \vdots & \ddots & \vdots \\ g_{r1} & \cdots & g_{rf} \end{bmatrix} \tag{2.40}$$

As before, the number of trials r should be chosen sufficiently large to ensure that the above matrix has reached its maximum rank.

Step 3

Let d be the nullity of the matrix of (2.40) less the number of independent multilinear identities found in step 1 for a given frame. Choose d many independent central identities from step 2. These together with the independent identities of step 1 form an independent generating set for all of the multilinear central identities of the given frame.

CHAPTER 3. MULTILINEAR IDENTITIES AND CENTRAL IDENTITIES OF DEGREE N $\leq$ 8

Preliminaries

We have used the procedure described in the previous chapter to obtain an independent generating set for all the multilinear identities and the multilinear central identities of each frame of degree $n < 9$ of the Q-algebra $M_3(Q)$ in terms of the e_iS_{ij}'s of the given frame. In other words, we will express the multilinear identities and central identities of a frame F_λ with f standard tableaus in the form:

$$\sum_{k=1}^{f} \alpha_k E_k{}^\lambda \tag{3.1}$$

where $\alpha_1, ..., \alpha_f \in Q$, and $E_k{}^\lambda = e_k{}^\lambda S_{k1}{}^\lambda$. For brevity, we will drop the superscript λ. Then one should remember that e_iS_{ij}'s of different frames are different. Furthermore, since in general, the e_i's are rather difficult to compute, we have used the relation $e_iS_{i1} = S_{i1}e_1$ to rewrite the multilinear identities and central identities as linear combination of the $S_{i1}e_1$'s of the given frame. Although it is preferred to express a multilinear identity or central identity in the more familiar form $\sum_{\pi \in S_n} \alpha_\pi \pi$, it is not always easy to do so. In general, this becomes a time consuming task for the multilinear identities and central identities of degree $n > 7$ in which the number of terms in each e_i is relatively large. In short, we will express the resulting multilinear identities and new multilinear central identities of each frame in the form:

$$\sum_{i=1}^{f} \alpha_i S_{i1} e_1 \tag{3.2}$$

In chapter 5, we will show that the union of the independent generating sets of the multilinear identities of all of the frames of degree n form an independent generating set for all of the multilinear identities of degree n.

Degree 6

There are no multilinear identities or multilinear central identities of degree $n < 6$. The first multilinear identity appears in degree six. This identity, which is given by (3.3), belongs to the eleventh representation, i.e., to the frame

$$F_{11} \quad : \quad \begin{array}{c} [\ ] \\ [\ ] \\ [\ ] \\ [\ ] \\ [\ ] \\ [\ ] \end{array}$$

and is given by:

$$I \cong S_{11} e_1 \tag{3.3}$$

S_{11} is the same as the identity permutation. So the above identity can be rewritten in the in the simplified form $\sum_{\pi \in S_6} sgn(\pi)\pi$ and is a well known identity. There are no central identities of degree 6.

Degree 7

There are six independent multilinear identities of degree 7 which together form an independent generating set for all of the multilinear identities of degree 7. Each of these identities can be obtained by expanding the identity (3.3) of degree 6. There are no multilinear central identities of degree 7. Below, we have listed the identities of degree 7 in the increasing frame order. The first of the independent identities belongs to the frame

$$
\begin{array}{lll}
F_{11} & : & [\ ][\ ][\ ] \\
 & & [\ ] \\
 & & [\ ] \\
 & & [\ ] \\
 & & [\ ]
\end{array}
$$

and is given by:

$$I \cong [-3S_{11} + 3S_{31} + 4S_{41} - 3S_{61} - 4S_{81} + 3S_{10,1} + 3S_{11,1} + S_{12,1} + 3S_{13,1} + S_{14,1} - 4S_{15,1}].e_1 \quad (3.4)$$

The second independent multilinear identity of degree 7 belongs to the thirteenth representation, i.e., the frame:

$$F_{13} \quad : \quad \begin{array}{l} [\ \ ][\ \ ] \\ [\ \ ][\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \end{array}$$

This identity is given by:

$$I \cong [4S_{11} + 5S_{21} + 6S_{31} - 5S_{51} + 4S_{61} + 5S_{91} + S_{10,1} - S_{11,1} + S_{12,1} - S_{13,1} - 4S_{14,1}].e_1 \tag{3.5}$$

The next three identities belong to the frame

$$F_{14} \quad : \quad \begin{array}{l} [\ \ ][\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \end{array}$$

and are as follows:

$$I \cong [S_{21} + S_{41}].e_1 \tag{3.6}$$

$$I \cong [S_{11} + S_{31} + S_{51}].e_1 \tag{3.7}$$

$$I \cong S_{61}e_1 \tag{3.8}$$

The last independent multilinear identity of degree 7 belongs to the frame:

$$F_{15} \quad : \quad \begin{array}{l} [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \\ [\ \ ] \end{array}$$

and is given by:

$$I \cong S_{11}e_1 \tag{3.9}$$

Degree 8

Identities

There are exactly 43 independent multilinear identities of degree 8. However, due to the length and number of these identities, we have decided to list only two of them. These two identities together with a new multilinear central identity which is mentioned later in this section form an independent generating set for all of the multilinear central identities of the representation 15, i.e., the frame

$$F_{15} \quad : \quad \begin{array}{l} [\ \ ][\ \ ][\ \ ] \\ [\ \ ][\ \ ] \\ [\ \ ][\ \ ] \\ [\ \ ] \end{array}$$

The two identities are given below by (3.10) and (3.11).

$$
\begin{aligned}
I \cong\ & [-224S_{11} - 422S_{21} + 132S_{31} - 252S_{41} + 55S_{51} - 440S_{61} - 280S_{71} - \\
& 201S_{81} - 24S_{91} + 13S_{10,1} + 29S_{11,1} - 246S_{12,1} - 313S_{13,1} - 60S_{14,1} + 143S_{15,1} + \\
& 223S_{16,1} - 20S_{17,1} + 249S_{18,1} + 52S_{19,1} + 115S_{20,1} - 37S_{21,1} - 49S_{22,1} - 215S_{23,1} + \\
& 194S_{24,1} - 77S_{25,1} - 219S_{26,1} + 72S_{27,1} + 96S_{28,1} + 81S_{29,1} + 188S_{30,1} - 67S_{31,1} + \\
& 17S_{32,1} - 15S_{33,1} - 75S_{34,1} + 50S_{35,1} + 107S_{36,1} + 37S_{37,1} + 301S_{38,1} + 413S_{39,1} + \\
& 34_{40,1} + 203S_{41,1} - 74S_{42,1} - 88S_{43,1} + 43S_{44,1} + 40S_{45,1} + 42S_{46,1} + 120S_{47,1} + \\
& 291S_{48,1} + 304S_{49,1} + 329S_{50,1} - 76S_{51,1} - 314S_{52,1} + 91S_{53,1} - 14S_{54,1} + \\
& 54S_{55,1} + 290S_{56,1} - 14S_{57,1} - 75S_{58,1} + 76S_{59,1} + 87S_{60,1} + 71S_{61,1} + 3S_{62,1} + \\
& 32S_{63,1} - 37S_{64,1} + 34S_{65,1} + 20S_{66,1} - 91S_{67,1} - 106S_{68,1} - 54S_{69,1}].e_1
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
I \cong\ & [-172S_{11} - 136S_{21} - 132S_{31} - 126S_{41} - 19S_{51} - 118S_{61} - 80S_{71} - \\
& 69S_{81} - 30S_{91} + 77S_{10,1} + 43S_{11,1} - 24S_{12,1} - 47S_{13,1} + 6S_{14,1} + 37S_{15,1} + 29S_{16,1} + \\
& 56S_{17,1} + 75S_{18,1} + 38S_{19,1} - 25S_{20,1} - 53S_{21,1} - 95S_{22,1} - 73S_{23,1} - 68S_{24,1} - \\
& 49S_{25,1} - 51S_{26,1} - 18S_{27,1} + 12S_{28,1} + 27S_{29,1} + 46S_{30,1} + 31S_{31,1} + S_{32,1} + 15S_{33,1} + \\
& 21S_{34,1} + 22S_{35,1} + 19S_{36,1} - S_{37,1} + 59S_{38,1} + 91S_{39,1} + 56S_{40,1} + 85S_{41,1} + \\
& 2S_{42,1} - 56S_{43,1} - 7S_{44,1} - 4S_{45,1} + 12S_{46,1} + 42S_{47,1} + 33S_{48,1} + 56S_{49,1} + 13S_{50,1} - \\
& 14S_{51,1} - 28S_{52,1} - S_{53,1} - 4S_{54,1} - 2S_{56,1} - 4S_{57,1} - 33S_{58,1} - 40S_{59,1} - 33S_{60,1} + \\
& S_{61,1} - 57S_{62,1} - 68S_{63,1} - 53S_{64,1} + 2S_{65,1} + 52S_{66,1} + S_{67,1} + 16S_{68,1} - 54S_{70,1}].e_1
\end{aligned}
\tag{3.11}
$$

The remaining multilinear identities, which are not mentioned here, are obtained by the exact same procedure as the other identities in this paper. Further computations show that all of these identities are consequences of the standard identity of degree 6. Thus every identity of degree < 9 is implied by $S_6(x_1, ..., x_6)$.

Central Identities

The multilinear central identities appear for the first time in degree 8. The first two multilinear central identities belong to the representation 14, i.e., to the frame F_{14} or:

```
[ ][ ][ ]
[ ][ ][ ]
[ ]
[ ]
```

These two central identities are given below by (3.12) and (3.13), and together they form an independent generating set for all the multilinear central identities of the frame F_{14} of degree 8.

$$I_c \cong [-407S_{11} - 518S_{21} - 814S_{41} + 444S_{61} + 259S_{71} - 222S_{81} - 259S_{91} + 444S_{10,1} + 407S_{11,1} + 259S_{12,1} + 333S_{13,1} - 481S_{14,1} - 222S_{15,1} + 111S_{16,1} + 629S_{17,1} + 1110S_{18,1} + 37S_{19,1} - 999S_{20,1} - 555S_{21,1} - 148S_{22,1} + 370S_{23,1} - 518S_{24,1} - 259S_{25,1} - 814S_{26,1} - 592S_{27,1} - 481S_{28,1} + 259S_{29,1} - 148S_{30,1} - 296S_{31,1} - 185S_{32,1} - 481S_{33,1} + 185S_{34,1} - 629S_{35,1} - 148S_{36,1} + 407S_{37,1} - 1147S_{38,1} - 925S_{39,1} + 222S_{40,1} - 555S_{41,1} + 333S_{42,1} + 259S_{43,1} - 148S_{44,1} - 1073S_{45,1} - 185S_{46,1} - 666S_{47,1} - 962S_{48,1} + 333S_{49,1} - 444S_{50,1} + 777S_{51,1} + 518S_{52,1} - 296S_{53,1} - 925S_{54,1} - 370S_{55,1}].e_1 \tag{3.12}$$

$$I_c \cong [-124S_{11} - 31S_{21} - 125S_{31} + 67S_{41} - 85S_{51} - 262S_{61} - 232S_{71} + 31S_{81} - 23S_{91} - 172S_{10,1} - 246S_{11,1} - 222S_{12,1} - 149S_{13,1} - 42S_{14,1} + 31S_{15,1} + 87S_{16,1} - 222S_{17,1} - 355S_{18,1} - 51S_{19,1} + 272S_{20,1} + 255S_{21,1} - 11S_{22,1} - 145S_{23,1} + 94S_{24,1} + 12S_{25,1} + 152S_{26,1} + 101S_{27,1} + 13S_{28,1} - 67S_{29,1} - 136S_{30,1} + 73S_{31,1} + 25S_{32,1} + 13S_{33,1} - 95S_{34,1} + 17S_{35,1} - 31S_{36,1} - 46S_{37,1} + 196S_{38,1} + 160S_{39,1} - 91S_{40,1} - 40S_{41,1} - 104S_{42,1} - 112S_{43,1} - 11S_{44,1} + 209S_{45,1} - 22S_{47,1} + 111S_{48,1} - 164S_{49,1} + 12S_{50,1} - 241S_{51,1} - 209S_{52,1} - 22S_{53,1} + 220S_{54,1} - 185S_{56,1}].e_1 \tag{3.13}$$

There are three independent multilinear central idetities in the fifteenth representation of degree 8, i.e., the frame:

```
[ ][ ][ ]
[ ][ ]
[ ][ ]
[ ]
```

However, since there are two independent identities in the frame F_{15}, according to the procedure, there is only one new multilinear central identity in the mentioned frame. Therefore, we state only one of the central identities of this frame. This new central identity, which is given by (3.14), together with the identities (3.10) and (3.11) forms an independent generating set for all of the multilinear central identities of the frame F_{15} of degree 8.

$$I_c \cong [322S_{11} + 340S_{21} + 438S_{31} + 612S_{41} + 169S_{51} - 380S_{61} - 16S_{71} -$$

$$
\begin{aligned}
&3S_{81} + 426S_{91} - 17S_{10,1} - 229S_{11,1} + 6S_{12,1} + 293S_{13,1} + 66S_{14,1} + 191S_{15,1} + \\
&49S_{16,1} - 140S_{17,1} + 339S_{18,1} - 122S_{19,1} + 211S_{20,1} + 11S_{21,1} + 143S_{22,1} + \\
&277S_{23,1} + 440S_{24,1} + 55S_{25,1} - 669S_{26,1} - 252S_{27,1} - 84S_{28,1} + 351S_{29,1} + \\
&182S_{30,1} - 37S_{31,1} - 97S_{32,1} - 375S_{33,1} - 471S_{34,1} - 136S_{35,1} + 47S_{36,1} + \\
&259S_{37,1} - 53S_{38,1} - 133S_{39,1} + 454S_{40,1} - 469S_{41,1} + 22S_{42,1} + 356S_{43,1} + 31S_{44,1} - \\
&314S_{45,1} - 30S_{46,1} + 84S_{47,1} + 93S_{48,1} + 400S_{49,1} + 197S_{50,1} - 154S_{51,1} - 308S_{52,1} + \\
&43S_{53,1} - 368S_{54,1} + 86S_{56,1} - 98S_{57,1} + 15S_{58,1} + 46S_{59,1} + 285S_{60,1} + 227S_{61,1} + \\
&453S_{62,1} + 710S_{63,1} + 119S_{64,1} - 410S_{65,1} - 238S_{66,1} - 43S_{67,1} - 634S_{68,1}].e_1
\end{aligned}
\tag{3.14}
$$

The last independent new central identity belongs to the frame F_{18} or:

[ ][ ]
[ ][ ]
[ ][ ]
[ ][ ]

and is given by:

$$
\begin{aligned}
I_c \;\cong\; &[4S_{11} + 21S_{21} + 12S_{31} + 21S_{41} + 7S_{51} + 21S_{61} + 8S_{71} + 21S_{81} + \\
&22S_{91} + 8S_{10,1} + 7S_{11,1} + 8S_{12,1} + 12S_{13,1} + 38S_{14,1}].e_1
\end{aligned}
\tag{3.15}
$$

As we shall prove in chapter 5, all the multilinear central identities of all of the frames of a given degree form an independent generating set for all the multilinear central identities of that degree. Therefore, the central identities (3.12)-(3.15) together with the identities (3.10) and (3.11) form an independent generating set for all of the multilinear central identities of degree 8.

CHAPTER 4. COMPLETE MULTILINEARIZATION

In this chapter, through the process of complete multilinearization we will prove that every identity of degree n of the ϕ-algebra $M_3(\phi)$ is implied by a set of multilinear identities, where ϕ is a field of characteristic zero or $p > 7$. We will also prove a similar result for the multilinear central identities. The concept of linearization and some of the major theorems of this chapter may also be found in "the Rings that are Nearly Associative" by Zhevlakov, Slin'ko, Shestakov and Shirshov [20].

Recall from linear algebra that the Vandermonde determinant is defined by:

$$V_n = \begin{vmatrix} 1 & \alpha_1 & \cdots & {\alpha_1}^{n-1} \\ 1 & \alpha_2 & \cdots & {\alpha_2}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \cdots & {\alpha_n}^{n-1} \end{vmatrix} \tag{4.1}$$

One can show by induction that the Vandermonde determinant of (4.1) simplifies to the following product:

$$V_n = \prod_{i<j} (\alpha_j - \alpha_i) \tag{4.2}$$

Theorem 4.1. Let $f = f(x_1, ..., x_m) \in \phi[X]$ be an identity of degree n of the ϕ-algebra R, where ϕ is a field of characteristic zero or prime $p > n$. Then every homogeneous component of f is also an identity of the ϕ-algebra R.

Proof. Let $f = f_0 + ... + f_k$, where each f_i is of degree i in x_1. Then k is the degree of x_1 in f, and we have $k \leq n$. Let $r_1, ..., r_m$ be arbitrary elements of the ring R. Since ϕ has characteristic zero or greater than n, we can choose $k+1$ distinct elements $\alpha_1, ..., \alpha_{k+1}$ from ϕ. For brevity, we let $f_i(r) = f_i(r_1, ..., r_m)$. Then for each $i = 1, ..., k+1$, we have:

$$f(\alpha_i r_1, r_2, ..., r_m) = f_0(r) + \alpha_i f_1(r) + {\alpha_i}^2 f_2(r) + ... + {\alpha_i}^k f_k(r) = 0$$

Hence:

$$\begin{bmatrix} 1 & \alpha_1 & \cdots & {\alpha_1}^k \\ 1 & \alpha_2 & \cdots & {\alpha_2}^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{k+1} & \cdots & {\alpha_{k+1}}^k \end{bmatrix} \begin{bmatrix} f_0(r) \\ f_1(r) \\ \vdots \\ f_k(r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let the matrix to the left be denoted by $[A]$, and the middle matrix be denoted by $[F_j(r)]$. Then we have:

$$[A]\,[F_j(r)] = [0]$$

The determinant of $[A]$ is the Vandermonde determinant V_{k+1} which can be simplified using equation (4.1). Since $\alpha_1, ..., \alpha_{k+1}$ are all distinct, this determinant is nonzero. Therefore, the matrix $[A]$ is nonsingular. Multiplying both sides of the above equation by $[A]^{-1}$, we get:

$$[F_j(r)] \;=\; 0$$

Therefore, $f_j(r_1, ..., r_m) = 0$ for any choice of $r_1, ..., r_m \in R$ and for each $j = 0, ..., k$. This implies that each f_i is an identity of the ϕ-algebrs R. Now we repeat the same process for the identities $f_0, ..., f_k$ with x_2, etc.. In the end, we have shown that every fragment which is homogeneous in each of the variables is an identity. Therefore, every homogeneous component of f is an identity.

□

We can modify the above theorem for central identities. But first we need the following lemma:

Lemma 4.2. Let $c_1, ..., c_k \in C(R)$. Then any linear combination of $c_1, ..., c_k$ is also in $C(R)$.

Proof. Let $\alpha_1, ..., \alpha_k$ be arbitrary elements of ϕ, and $x \in R$. Then we need to show that $\alpha_1 c_1 + ... + \alpha_k c_k$ satisfies the conditions (2.4)–(2.7). Using equation (2.2), distributivity, and the fact that $c_1, ..., c_k \in C(R)$, we get:

$$\begin{aligned}
(\alpha_1 c_1 + ... + \alpha_k c_k)x &= (\alpha_1 c_1)x + ... + (\alpha_k c_k)x \\
&= \alpha_1(c_1 x) + ... + \alpha_k(c_k x) \\
&= \alpha_1(x c_1) + ... + \alpha_k(x c_k) \\
&= x(\alpha_1 c_1) + ... + x(\alpha_k c_k)
\end{aligned}$$

$$= x(\alpha_1 c_1) + ... + \alpha_k c_k)$$

So $\alpha_1 c_1 + ... + \alpha_k c_k$ satisfies the condition (2.4). Similarly, we can show that the associativity conditions (2.5), (2.6), and (2.7) are satisfied.

□

Theorem 4.3. Let $f = f(x_1, ..., x_m) \in \phi[X]$ be a central identity of degree n of the ϕ-algebra R, where ϕ is a field of characteristic zero or prime $p > n$. Then every homogeneous component of f is either an identity or a central identity of the ϕ-algebra R. Furthermore, at least one of the homogeneous components must be a central identity.

Proof. The proof is very similar to the proof of Theorem 4.1. As before, we break f into the sum of polynomials $f_0, ..., f_k$, where the degree of x_1 in f_i is i. Let $\alpha_1, ..., \alpha_{k+1}$ be distinct elements of ϕ. Then for arbitrary $r_1, ..., r_m \in R$ and for each $i = 1, ..., k+1$, we have:

$$f(\alpha_i r_1, r_2, ..., r_m) = f(r) + \alpha_i f_1(r) + {\alpha_i}^2 f_2(r) + ... + {\alpha_i}^k f_k(r) \in C(R)$$

Thus the following holds for some $c_0, ..., c_k \in C(R)$:

$$\begin{bmatrix} 1 & \alpha_1 & \cdots & {\alpha_1}^k \\ 1 & \alpha_2 & \cdots & {\alpha_2}^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{k+1} & \cdots & {\alpha_{k+1}}^k \end{bmatrix} \begin{bmatrix} f_0(r) \\ f_1(r) \\ \vdots \\ f_k(r) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{bmatrix}$$

or:

$$[A][F_j(r)] = [c_0 \dots c_k]^T$$

Since, as in Theorem 4.1, $[A]$ is nonsingular, we can multiply both sides of the above system by $[A]^{-1}$ to get:

$$[F_j(r)] = [A]^{-1}[c_0 \dots c_k]^T$$

By Lemma 4.2 a linear combination of elements of $C(R)$ is also in the center of R. Therefore, $f_j(r) \in C(R)$ for $j = 0, \dots, k$. Since r_i's are arbitrary, the above equation implies that each f_i is either an identity or a central identity. As in Theorem 4.1, we continue the process to get the result. Obviously, at least one of the homogeneous components must be a central identity. Otherwise, f is an identity which is a contradiction.

$\square$

Let $f = f(x_1, \dots, x_m) \in \phi[X]$ be a nonassociative polynomial and $y_1, \dots, y_k \in X - \{x_1, \dots, x_m\}$. Then for each $i = 1, \dots, m$ we define a polynomial $f\mathbf{L}_i^k \in \phi[X]$ by the following formula. The notation $\overline{x_i}$ indicates that the coefficient of x_i is zero, i.e., $\overline{x_i} = 0.x_i = 0$.

$$\begin{aligned}
&f\mathbf{L}_i^k(x_1, \dots, x_{i-1}, y_1, y_2, \dots, y_k, x_{i+1}, \dots, x_m) \\
&= f(x_1, \dots, x_{i-1}, y_1 + y_2 + \dots + y_k, x_{i+1}, \dots, x_m) \\
&- \textstyle\sum_{q=1}^k f(x_1, \dots, x_{i-1}, y_1 + \dots + \overline{y_q} + \dots + y_k, x_{i+1}, \dots, x_m) \\
&+ \textstyle\sum_{1 \le q_1 \le q_2 \le k} f(x_1, \dots, x_{i-1}, y_1 + \dots + \overline{y_{q_1}} + \dots + \overline{y_{q_2}} + \dots + y_k, x_{i+1}, \dots, x_m) \\
&- \dots
\end{aligned}$$

$$+(-1)^{k-1}\sum_{q=1}^{k} f(x_1,...,x_{i-1},y_q,x_{i+1},...,x_m) \tag{4.3}$$

Obviously, if f is an identity of the ϕ-algebra R, then so is $f\mathbf{L}_i^k$. It also follows from the definition that $\mathbf{L}_i^k$ is a linear operator, i.e.,

$$(f_1+f_2)\mathbf{L}_i^k \;=\; f_1\mathbf{L}_i^k + f_2\mathbf{L}_i^k$$

Lemma 4.4. Let $g : R\times R\times R\times ...\times R$ be a function on n variables defined for a ϕ-algebra R. Suppose g is linear on each of its components. Then for any $r_1,...,r_k \in R$, where $n \le k$, we have:

$$\begin{aligned}
&g(r_1+r_2+...+r_k,...,r_1+r_2+...+r_k)\\
&-\textstyle\sum_{q=1}^{k} g(r_1+...+\overline{r_q}+...+r_k,...,r_1+...+...+\overline{r_q}+...+r_k)\\
&+\ \textstyle\sum_{1\le q_1\le q_2\le k} g(r_1+...+\overline{r_{q_1}}+...+\overline{r_{q_2}}+...+r_k,...,r_1+...+\overline{r_{q_1}}+...+\overline{r_{q_2}}+...+r_k)\\
&+\ .\ .\ .\\
&+\ (-1)^{k-1}\ \textstyle\sum_{q=1}^{k} g(r_q,r_q,...,r_q)
\end{aligned}$$

$$= \begin{cases} \sum_{(i_1...i_n)\in S_n} g(r_{i_1},r_{i_2},...,r_{i_n}) & \text{if } n = k \\ 0 & \text{if } n < k \end{cases}$$

Proof. Because of the linearity of the function g, the left side of the equality to be proved can be rewritten as a linear combination of the elements of the form $g(r_{j_1},...,r_{j_n})$ with integer coefficients. Let s be the number of r_{i_l}'s in $g(r_{i_1},...,r_{i_n})$

which are different. Then the coefficient for $g(r_{j_1},...,r_{j_n})$ is

$$1-\binom{k-s}{1}+\binom{k-s}{2}-...+(-1)^{k-s}\binom{k-s}{k-s}$$

We consider the following two cases for each element $g(r_{j_1},...,r_{j_n})$.

Case 1: Suppose $n<k$. Since $s\leq n$, we must have $s<k$, and by binomial theorem the above alternating sum is $(1-1)^{k-s}$ which is zero.

Case 2: Suppose $n=k$. If $s<k$, then the coefficient of g is zero as we showed in the first case. Otherwise, we have $s=n=k$ in which case the coefficient of g is obviously 1.

$\square$.

The following theorem establishes an easy way of computing $f\mathbf{L}_i^k$ for a nonassociative polynomial f.

Theorem 4.5. Let $f(x_1,...,x_m)$ be a monomial of degree n in x_i and $g = g(x_1,...,x_{i-1},\ y_1,y_2,...,y_n,x_{i+1},...,x_m)$ be a monomial linear in $y_1,...,y_n \in X-\{x_1,...,x_m\}$ such that

$$f(x_1,...,x_m) \ = \ g(x_1,...,x_{i-1},\ x_i,x_i,...,x_i,x_{i+1},...,x_m)$$

Then the following relation holds:

$$f\mathbf{L}_i^k(x_1,\dots,x_{i-1},y_1,\dots,y_k\ ,x_{i+1},\dots,x_m)$$

$$= \begin{cases} \sum_{(i_1,...,i_n)\in S_n} g(x_1,\dots,x_{i-1},y_{i_1},\dots,y_{i_n}\,x_{i+1},\dots,x_m) & \text{if} \quad n=k \\ 0 & \text{if} \quad n<k \end{cases}$$

Proof. We fix $x_1, ..., x_{i-1}, x_{i+1}, ..., x_m$ in g. Then g can be considered as a monomial in n variables. Since by the hypothesis g is linear in each of its components, the conclusion of Lemma 4.4 should hold for g. But since by assumption of the thorem $f(x_1, ..., x_m) = g(x_1, ..., x_{i-1}, x_i, x_i, ..., x_i, x_{i+1}, ..., x_m)$, the right side of the equation equals $f\mathbf{L}_i^k$ by the definition of the $f\mathbf{L}_i^k$ operator and Lemma 4.4.

□

Example. Let $f = [\ x_2{}^2(x_1x_2)\ ]x_3{}^2$. Then:

$$\begin{aligned}
f\mathbf{L}_1^2 &= 0 \\
f\mathbf{L}_1^1 &= [\ x_2{}^2(y_1x_2)\ ]x_3{}^2 \\
f\mathbf{L}_1^1\mathbf{L}_2^3 &= [\ (z_1z_2)(y_1z_3)\ ]x_3{}^2 + [\ (z_2z_1)(y_1z_3)\ ]x_3{}^2 \\
&+ [\ (z_1z_3)(y_1z_2)\ ]x_3{}^2 + [\ (z_3z_1)(y_1z_2)\ ]x_3{}^2 \\
&+ [\ (z_2z_3)(y_1z_1)\ ]x_3{}^2 + [\ (z_3z_2)(y_1z_1)\ ]x_3{}^2
\end{aligned}$$

and

$$\begin{aligned}
f\mathbf{L}_1^1\mathbf{L}_2^3\mathbf{L}_3^2 &= [\ (z_1z_2)(y_1z_3)\ ](u_1u_2) + [\ (z_2z_1)(y_1z_3)\ ](u_1u_2) \\
&+ [\ (z_1z_3)(y_1z_2)\ ](u_1u_2) + [\ (z_3z_1)(y_1z_2)\ ](u_1u_2) \\
&+ [\ (z_2z_3)(y_1z_1)\ ](u_1u_2) + [\ (z_3z_2)(y_1z_1)\ ](u_1u_2) \\
&+ [\ (z_1z_2)(y_1z_3)\ ](u_2u_1) + [\ (z_2z_1)(y_1z_3)\ ](u_2u_1) \\
&+ [\ (z_1z_3)(y_1z_2)\ ](u_2u_1) + [\ (z_3z_1)(y_1z_2)\ ](u_2u_1) \\
&+ [\ (z_2z_3)(y_1z_1)\ ](u_2u_1) + [\ (z_3z_2)(y_1z_1)\ ](u_2u_1)
\end{aligned}$$

If f is a homogeneous polynomial of the type $[k_1, ..., k_m]$, then the multilinear polynomial $f\mathbf{L}_1^{k_1}\mathbf{L}_2^{k_2}...\mathbf{L}_m^{k_m}$ is called the complete multilinearization of the polynomial f. For instance, The multilinear polynomial $f\mathbf{L}_1^1\mathbf{L}_2^3\mathbf{L}_3^2$ in the above example is the complete multilinearization of the polynomial $f = [\ {x_2}^2(x_1x_2)\ ]{x_3}^2$. Now, we shall prove that under certain conditions every homogeneous identity $f(x_1, ..., x_m)$ of the type $[k_1, ..., k_m]$ is equivalent to the multilinear identity $f\mathbf{L}_1^{k_1}\mathbf{L}_2^{k_2}...\mathbf{L}_m^{k_m}$.

Theorem 4.6. Let ϕ be a field of characteristic zero or $p > n$, and $f = f(x_1, ..., x_m) \in \phi[X]$ be a homogeneous identity of degree $\leq n$ of the ϕ-algebra R. Then f is equivalent to a multilinear identity of degree n.

Proof. Suppose f has the type $[k_1, ..., k_m]$. Then we completely linearize f by inductively constructing $g = f\mathbf{L}_1^{k_1}\mathbf{L}_2^{k_2}...\mathbf{L}_m^{k_m}$ as follows:

$$f\mathbf{L}_1^{k_1}(y_{11}, ..., y_{1k_1}, x_2, ..., x_m)$$

$$f\mathbf{L}_1^{k_1}\mathbf{L}_2^{k_2}(y_{11}, ..., y_{1k_1}, y_{21}, ..., y_{2k_2}, x_3, ..., x_m)$$

$$\vdots$$

$$f\mathbf{L}_1^{k_1}...\mathbf{L}_m^{k_m}(y_{11}, ..., y_{1k_1}, ..., y_{m1}, ..., y_{mk_m})$$

We know that the complete linearization of f which is given by g is a multilinear identity. To show that g implies f, we let $y_{ij} = x_i$ in g for $j = 1, ..., k_i$ and each $i \leq m$. Then we get the following identity:

$$(k_1!k_2!...k_m!)f(x_1, ..., x_m)$$

Since the characteristic of ϕ is a prime $p > n$ and $n > k_i$ for $i = 1, ..., m$, p does not divide $k_1!, ..., k_m!$. Thus the product $k_1!\ k_2!...k_m!$ is not a multiple of p and is a nonzero element of the field ϕ, and therefore, it is invertible. Now, we can multiply g by the inverse of $k_1!...k_m!$ to get f. Hence g also implies f, and therefore, the two identities are equivalent.

□

If we replace 'identity' by 'central identity' in the proof of the above theorem, all the statements remain valid. Therefore, we have the following important result for the central identities:

Theorem 4.7. Let ϕ be field of characteristic $p > n$, and $f = (x_1, ..., x_m) \in \phi[X]$ be a homogeneous central identity of degree $\leq n$ of the ϕ-algebra R. Then f is equivalent to a multilinear central identity of degree n.

Now, we are ready to state and prove two theorems which are extremely important to our method of computing the identities and central identities used in the procedures mentioned in the first chapter.

Theorem 4.8. Let ϕ be a field of characteristic zero or $p > n$, and $f = f(x_1, ..., x_m) \in \phi[X]$ be an identity of degree $\leq n$ of the ϕ-algebra R. Then f is implied by a set of multilinear identities.

Proof. Let $f_1, ..., f_k$ be the homogeneous components of f. Since the characteristic of ϕ is zero or greater than p, by Theorem 4.1, f_i is an identity of the ϕ -algebra R for each $i = 1, ..., k$. On the other hand, by Theorem 4.6, every homogeneous component of f is equivalent to a multilinear identity. It follows by definition that f is implied by these multilinear identities.

□

Similarly, we can use Theorems 4.3 and 4.7 to obtain the following important theorem for the central identities:

Theorem 4.9 Let ϕ be a field of characteristic $p > n$, and $f = f(x_1, ..., x_m) \in \phi[X]$ be a central identity of degree $\leq n$ of the ϕ-algebra R. Then f is implied by a set consisting of multilinear identities and central identities.

Now, let us sum up the results of this chapter by relating the last two theorems to the procedures described in the second chapter. As we shall prove in the next chapter, the procedures give us a method of computing all the multilinear identities and central identities of degree no greater than 8 of the ϕ-algebra $M_3(\phi)$, where ϕ is either the field of all rationals or Z_p with prime $p > 7$. But according to the above theorems, every identity (central identity) of the ϕ-algebra $M_3(\phi)$ is implied by a set of multilinear identities (multilinear central identities). So it is sufficient to find only the multilinear identities and the multilinear central identities, and that is exactly what the procedure does. In fact, as one may notice, the procedure is valid for identities and central identities of higher degrees. However, as we mentioned before, this may be too much computation for current computers.

CHAPTER 5. PROOF OF THE PROCEDURE

Fundamental Theorems for Tableaus

Let S_n be the group of all permutations on a set with n objects and F be a field of characteristic zero or $p > n$. Consider all the sums of the form:

$$a = \sum_{s \in S_n} \alpha(s).s = \sum_{s \in S_n} s.\alpha(s) \tag{5.1}$$

with arbitrary numerical coefficients $\alpha(s) \in \phi$.

The collection of all sums of the form (5.1) is called the group ring or the group algebra over the symmetric group S_n and is denoted by O_{S_n} or simply O. By definition, the group elements commute with the numbers and form a basis for the group ring. An element a of the group ring is called idempotent if $a^2 = a$. The element a of the group ring is called essentially idempotent if $a^2 = \lambda a$ for some nonzero constant λ. If a is essentially idempotent, then $\frac{a}{\lambda}$ is idempotent. The multiplication of the group ring elements is defined by:

$$\{\sum_{s \in S_n} \alpha(s).s\} \{\sum_{\pi \in S_n} \alpha(\pi).\pi\} = \sum_{s,\pi \in S_n} \alpha(s)\alpha(\pi).s\pi \tag{5.2}$$

The ultimate goal is to express the group ring O as a direct sum of subrings $M_1, ..., M_k$, where each M_i is isomorphic to a full matrix ring over ϕ. Indeed, for the purpose of this paper, we only need to find an appropriate basis for the group ring. This is achieved by breaking the group ring into subrings using tableaus. Most of the elementary tableau theorems proved in this section were originally developed by Young and his students. Some of these theorems may also be found in "Representation of the Groups" by Herman Boerner [6]. Throughout the rest of this section, we assume that the tableau T belongs to the frame F and the tableau T' belongs to the frame F'.

Theorem 5.1. Let T and T' be two tableaus of the same degree but different frames. If $F > F'$, then there exist two numbers in one row of T that appear in one column of T'.

Proof. Let T be a $(m_1, ..., m_r)$ and T' be a $(m_1', ..., m_k')$ tableau. Suppose no two numbers in the first row of T occur in one column of T'. This implies that $m_1 \leq m_1'$. Since $F > F'$, we must have $m_1 = m_1'$. One may note that since no two numbers in one row of T occur in one column of T', the fact remains the same if any column operation is applied to T'. Thus we can apply a column operation to T' to move all the numbers occuring in the first row of T to the first row of T'. Now, we can show that $m_2 = m_2'$ and so on. Therefore, we have $F = F'$ which is a contradiction.

□

Theorem 5.1 may be rephrased as follows.

Corollary 5.2. Let T and T' be two tableaus of the same degree such that

$F < F'$. Then there exist two numbers in one column of T that appear in one row of T'.

Theorem 5.3. Let T and T' be two tableaus of degree n from different frames. Then $\overline{e}.\overline{e'} = 0$.

Proof. We have to show that $PQ.P'Q' = 0$. We consider two cases. If $F < F'$, we show that the inner pair makes the product zero because QP' is zero. If $F > F'$, we show that the outer pair makes the product zero because $P\pi Q'$ is zero $\forall \pi \in S_n$.
Case 1: Assume $F < F'$. Then by Corollary 5.2, there exist two numbers in one column of T that occur in one row of T'. Let t be the transposition that interchanges these two numbers. Then t is simultaneously a vertical permutation for T and a horizontal permutation for T'. Using Theorem 2.2 and the fact that $t^2 = I$, we get:

$$\begin{aligned}
\overline{e}.\overline{e'} &= PQ.P'Q' \\
&= PQ.I.P'Q' \\
&= PQ.t^2.P'Q' \\
&= PQt.tP'Q' \\
&= -PQ.P'Q' \\
&= -\overline{e}.\overline{e'}
\end{aligned}$$

Therefore, we have:

$$\overline{e}.\overline{e'} = 0$$

Case 2: Assume $F > F'$. Given any permutation $\pi \in S_n$, we have:

$$\begin{aligned} P\pi Q' &= P\pi Q'.\pi^{-1}\pi \\ &= P.\pi Q'\pi^{-1}.\pi \end{aligned}$$

By Corollary 2.3, $\pi Q'\pi^{-1}$ is another Q'' for a tableau T'' belonging to the same frame. Let $\pi Q'\pi^{-1} = Q''$. Then:

$$P\pi Q' = PQ''\pi$$

T'' belongs to the frame F', and by assumption $F > F'$. Therefore, by Theorem 5.1, there exist two numbers in one row of T that occur in one column of T''. Once again, let t be the transposition that interchanges these two numbers. Therefore, for any $\pi \in S_n$, we have:

$$\begin{aligned} P\pi Q' &= PQ''\pi \\ &= P.I.Q''\pi \\ &= P.t^2.Q''\pi \\ &= Pt.tQ''\pi \\ &= -PQ''\pi \\ &= -P\pi Q' \end{aligned}$$

The above implies that $P\pi Q' = 0 \quad \forall \pi \in S_n$. On the other hand, we know that $QP' = \sum_{\pi \in S_n} \lambda_\pi \pi$ for some coefficients $\lambda_\pi \in \phi$. Consequently, we have:

$$\begin{aligned} \bar{e}.\overline{e'} &= PQ.P'Q' \\ &= P.QP'.Q' \end{aligned}$$

$$\begin{aligned} &= P\{\sum_{\pi \in S_n} \lambda_\pi \pi\} Q' \\ &= \sum_{\pi \in S_n} \lambda_\pi \{P \pi Q'\} \\ &= 0 \end{aligned}$$

□

Theorem 5.4. If $pqT = T'$, then no two numbers in one row of T occur in one column of T'.

Proof As we mentioned in chapter 2, T' can be obtained by first applying a horizontal permutation p to T, and then applying a vertical permutation pqp^{-1} to the tableau pT. Let x_1 and x_2 be any two numbers in one row of T. Then p leaves x_1 and x_2 in the same row. Since pqp^{-1} is a vertical permutation for pT, it leaves these two numbers in their respective columns. Therefore, they appear in different columns of the tableau pqT.

□

Theorem 5.5. Suppose no two numbers which occur in one row of T occur in one column of πT. Then π is a pq for T.

Proof. The numbers in the first column of πT appear in different rows of T. Thus we can apply a horizontal permutation to T move them to the first column. Keeping the first column of this new tableau fixed, we can apply another horizontal permutation to the resulting tableau to move the numbers of the second column of

πT to the second column of the new tableau and so on. So we can apply a horizontal permutation p to T to move all the numbers to their correct columns in πT. Then we can apply a column operation to move them to their appropriate boxes. Thus π is a pq for T.

$\square$

Similarly, we can prove the following theorem:

Theorem 5.6 Suppose no two numbers which occur in one column of T appear in one row of πT. Then π is a qp for T.

Theorem 5.7 Let T_i and T_j be any two standard tableaus from the same frame such that $i > j$. Then there exist two numbers in one column of T_i that occur in one row of T_j.

Proof. Suppose the two tableaus are different for the first time at the intersection of the k–th row and the r–th column, i.e., at the position $(k\ ,\ r)$. Let x and y be the two numbers occupying the $(k\ ,\ r)$–th position of the tableaus T_i and T_j respectively. Since $T_i > T_j$, we must have $x > y$. We note that since the numbers in the first column of a standard tableau are uniquely determined by the numbers in the previous positions, and both tableaus are identical prior to the (k , r)–th position, we must have $r > 1$. Now, we like to determine the position of y in T_i. Suppose y occupies the position $(m\ ,\ n)$ in T_i. Since $(k\ ,\ r)$ is the first position at which the two tableaus are different, our choices are restricted to the positions of the form $(m = k \ , \ n > r)$ and $(m > k \ , \ n)$. Furthermore, since T_i is a standard tableau and $x > y$, the first choice is out of question, and for the second choice we must have $n < r$. Hence the only possiblities are the positions of the form $(m > k \ , \ n < r)$, i.e., to the left and below $(k \ , \ r)$. Thus we have the following picture for the two tableaus T_i and T_j :

$$
\begin{array}{rccccccc}
 & & & n: & & r: & \\
 & [\ \ ] & \cdots & [\ \ ] & \cdots & [\ \ ] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
k: & [\ \ ] & \cdots & [\,z\,] & \cdots & [\,x\,] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
m: & [\ \ ] & \cdots & [\,y\,] & \cdots & [\ \ ] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
 & & & T_i & & &
\end{array}
\qquad
\begin{array}{rccccccc}
 & & & n: & & r: & \\
 & [\ \ ] & \cdots & [\ \ ] & \cdots & [\ \ ] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
k: & [\ \ ] & \cdots & [\,z\,] & \cdots & [\,y\,] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
m: & [\ \ ] & \cdots & [\ \ ] & \cdots & [\ \ ] & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
 & & & T_j & & &
\end{array}
$$

Since the first time the two tableaus are different is at the $(k \ , \ r)$–th position, they must have the same number at the $(k \ , \ n)$–th position. This is indicated by letter z in the above diagram. Therefore, the two numbers y and z that appear in the n–th column of T_i occur in one row of the tableau T_j, namely the k–th row.

$\square$

Corollary 5.8. Let T_i and T_j be standard tableaus from the same frame such that $i > j$. Then $\bar{e}_i \bar{e}_j = 0$.

Proof. By Theorem 5.7, there exists two numbers in one column of T_i that occur in one row of T_j. Then the transposition t that interchanges these two numbers is simultaneously a 'q' for T_i and a 'p' for T_j. So we have:

$$
\begin{aligned}
\overline{e}_i\overline{e}_j &= P_iQ_iP_jQ_j \\
&= P_iQ_i.I.P_jQ_j \\
&= P_iQ_i.t^2.P_jQ_j \\
&= P_i.Q_it.tP_j.Q_j \\
&= -P_iQ_iP_jQ_j \\
&= -\overline{e}_i\overline{e}_j
\end{aligned}
$$

Therefore, $\overline{e}_i\overline{e}_j = 0$.

$\square$

Theorem 5.9. (Van Neumann). Let $\overline{e} = \sum_{pq} sgn(q).pq$ for a given tableau. Then $\overline{e}^2 = \lambda\overline{e}$ for some constant λ.

Proof. Using Theorem 2.2, we get:

$$
\begin{aligned}
p\overline{e}^2q &= pPQPQq \\
&= sgn(q).PQPQ \\
&= sgn(q)\overline{e}^2 \qquad \forall p \in \overline{P},\ q \in \overline{Q}
\end{aligned}
$$

Now, let. $\overline{e}^2 = \sum_{s\in S_n} \alpha(s).s$. We want to find the coefficient of each permutation $\pi \in S_n$ in the summation. We consider the following two cases.

Case 1: Suppose $\pi = pq$. Multiplying $\overline{e}^2$ on the left by p and on the right by q, we get:

$$
p\overline{e}^2q = p\{\sum_s \alpha(s).s\}q
$$

or:

$$sgn(q)\sum_s \alpha(s).s \;=\; \sum_s \alpha(s).psq$$

We note that each element of S_n appears exactly once on each side. Therefore, to find the coefficient of pq we may let $s = pq$ on the left side and $s = I$ on the right. Then we get $sgn(q).\alpha(pq) = \alpha(I)$ or $\alpha(pq) \;=\; sgn(q).\alpha(I)$.

Case 2: π is not a pq. Then by Theorem 5.4, there exist two numbers in one row of T that occur in one column of πT. Let t be the transposition that interchanges these two numbers. Then t is a row operation for T and $\pi^{-1}t\pi$ is a column operation for T. Let $p = t$ and $q = \pi^{-1}t\pi$, we get:

$$\sum_s \alpha(s).t.s.\pi^{-1}t\pi \;=\; sgn(\pi^{-1}t\pi)\sum_s \alpha(s).s$$

To find the coefficient of π, we let $s = \pi$ in the above. Then:

$$\alpha(\pi)\pi \;=\; sgn(q)\alpha(\pi)\pi$$

which implies that $\alpha(\pi) = -\alpha(\pi)$ or $\alpha(\pi) \;=\; 0$.

Putting the results of the first and the second case together, we get:

$$\begin{aligned}\bar{e}^2 &= \sum_s \alpha(s).s \\ &= \sum_{pq} sgn(q)\alpha(I)pq \\ &= \alpha(I)\bar{e}\end{aligned}$$

Now $\bar{e}^2 = \lambda\bar{e}$ for $\lambda = \alpha(I)$.

□

Theorem 5.10. Let $\bar{e} \;=\; \sum_{pq} sgn(q).pq$ belong to a tableau T. Then $\bar{e}$ is essentially idempotent.

Proof. In the previous theorem, we showed that $\bar{e}^2 = \lambda\bar{e}$, where λ is the coefficient of the identity permutation in $\bar{e}^2$, i.e., $\lambda = \alpha(I)$. So it remains to show that $\lambda \neq 0$. The group ring O is a vectorspace with $n!$ basis elements, say $\pi_1 = I$, $\pi_2, ..., \pi_{n!}$. Let π act as a linear transformation on O by right multiplication. Then the trace of the identity transformation is $Tr(I) = n!$, and the trace of any other permutation is 0. Now, we pick a basis for $O\bar{e}$, say $\pi_1\bar{e}, ..., \pi_d\bar{e}$, where d is the degree of $O\bar{e}$. Extend this basis to a basis of O. Let $\bar{e}$ act as a linear transformation on the right. Taking advantage of the fact that trace is invariant under the change of basis, we get:

$$\begin{aligned} Tr(\bar{e}) &= Tr\{\sum_{pq} sgn(q).pq\} \\ &= \sum_{pq} sgn(q).Tr(pq) \\ &= sgn(I)Tr(I) \\ &= Tr(I) \\ &= n! \end{aligned}$$

On the other hand, since by Theorem 5.9, $\pi_i\bar{e}^2 = \lambda\pi_i\bar{e}$ for $i = 1, ..., d$, we have:

$$Tr(\bar{e}) = \lambda d$$

Therfore, $\lambda d = n!$ which implies that λ is nonzero and $\lambda = \dfrac{n!}{d}$.

□

Remark 5.11. Given $\bar{e}_i = \sum_{pq} sgn(q)pq$ for a tableau T_i, then $e = \dfrac{d}{n!}\bar{e}$

is obviously an idempotent element of the groupring. This is a very important result which is used repeatedly in the remaining of this paper. Furthermore, since e_i is just a nonzero scalar multiple of $\overline{e}_i$, Theorem 5.3 and Corallory 5.8 are also valid if $\overline{e}_i$ is replaced by e_i.

Multiplication of the Group Ring Elements

In this section, we will introduce a very useful and simple way of multiplying the group ring elements which belong to the same frame. The method computes the product of two groupring elements $a = \sum_{i,j}^{f,f} \alpha_{ij} e_i S_{ij}$ and $b = \sum_{i,j}^{f,f} \beta_{ij} e_i S_{ij}$, where $e_i S_{ij}$'s belong to the same frame, using the multiplication of matrices. Note that the idempotency of the e_i's and equation (2.17) are frequently used throughout the rest of this chapter. With the help of the following two theorems we can multiply the $e_i S_{ij}$'s and e_i's of the same frame.

Theorem 5.12. Let T_i and T_j belong to the same frame. Then $e_i e_j = \varepsilon_{ij} e_i S_{ij}$, where

$$\varepsilon_{ij} = \begin{cases} sgn(q) & \text{if} \quad S_{ji} = qp \ \text{(for } T_i\text{)} \\ 0 & \text{otherwise} \end{cases}$$

Proof. We consider two cases.

Case 1: The permutation S_{ji} that takes T_i to T_j is a qp for T_i. Then we have:

$$\begin{aligned} e_i e_j &= e_i . S_{ji} e_i S_{ij} \\ &= P_i Q_i . qp P_i Q_i . S_{ij} \end{aligned}$$

$$
\begin{aligned}
&= P_i.Q_i q.p P_i.Q_i S_{ij} \\
&= sgn(q).P_i Q_i P_i Q_i.S_{ij} \\
&= sgn(q) e_i^2 S_{ij} \\
&= sgn(q) e_i S_{ij}
\end{aligned}
$$

Case 2: The permutation S_{ji} is not a qp for T_i. Then by theorem 5.6, there exist two numbers in one column of T_i that occur in one row of T_j. Let t be the transposition that interchanges these two numbers. t is obviously a vertical permutation for T_i and a horizontal permutation for T_j. So we have:

$$
\begin{aligned}
e_i e_j &= e_i.I.e_j \\
&= e_i.t^2.e_j \\
&= P_i Q_i t.t P_j Q_j \\
&= -P_i Q_i.P_j Q_j \\
&= -e_i e_j
\end{aligned}
$$

Therefore, if S_{ji} is not a qp for T_i, then $e_i e_j = 0$ which completes the proof.

□

Corollary 5.13. Let T_i and T_j belong to the same frame. Then $e_i S_{ij}.e_k S_{kl} = \varepsilon_{jk} e_i S_{il}$.

Proof. The result is directly obtained using the product rule established in the previous theorem:

$$e_i S_{ij}.e_k S_{kl} = S_{ij} e_j.e_k S_{kl}$$

$$
\begin{aligned}
&= S_{ij}.e_j e_k.S_{kl} \\
&= S_{ij}.\varepsilon_{jk} e_j S_{jk}.S_{kl} \\
&= \varepsilon_{jk}.S_{ij} e_j.S_{jk} S_{kl} \\
&= \varepsilon_{jk}.e_i S_{ij}.S_{jk} S_{kl} \\
&= \varepsilon_{jk} e_i.S_{ij} S_{jl} \\
&= \varepsilon_{jk} e_i S_{il}
\end{aligned}
$$

□

According to the above corollary, the product of elements of the form $\sum \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda$ is of the same form. Therefore, the set of all the elements of the group ring that can be expressed as a linear combination of the $e_i S_{ij}$'s of a fixed frame are closed under the group ring multiplication. We state this in the following theorem.

Theorem 5.14. Given a frame F_λ, all the elements of the form $\sum \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda$ form a subring of the group ring O. We call this subring F^λ.

Multiplication Rule

We will now show that the subrings F^λ are matrix like rings. Since we will be dealing with products of matrices and simultaneously with the products of F^λ as subrings of the group ring, we will use ' $*$ ' as the multiplication in F^λ.

Let $A = [\alpha_{ij}]_{f \times f}$ be a matrix with entries from the field ϕ, and F_λ be a fixed frame with f standard tableaus. We map A to $\overline{A} = \sum_{i,j}^{f,f} \alpha_{ij} e_i S_{ij} \in F^\lambda$. So the ij-th entry of the matrix A is the coefficient of $e_i S_{ij}$ in $\overline{A}$. Thus we map the matrix A to a group ring element $\overline{A}$. Also let E_{ij} be the unit matrix with a 1 at the ij–th position and zeros elsewhere. Using Corollary 5.13, we get:

$$\begin{aligned}
\overline{E_{ij}} * \overline{E_{kl}} &= e_i S_{ij}.e_k S_{kl} \\
&= \varepsilon_{jk} e_i S_{il} \\
&= \overline{E_{ij}\ (\varepsilon_{ij})\ E_{kl}}
\end{aligned}$$

where the ij–th entry of the $f \times f$ matrix (ε_{ij}) is simply ε_{ij}. Therefore, we have:

$$\overline{A} * \overline{B} = \overline{A\,(\varepsilon_{ij})\,B} \tag{5.3}$$

We usually refer to (ε_{ij}) by A_I. One should note that on the left side we have the product of the elements of the group ring, while $A\,(\varepsilon_{ij})\,B$ refers to the multiplication of matrices and is easily computed.

If T_i and T_j are two standard tableaus from the same frame such that $i > j$, then by Theorem 5.7 there exists two numbers in one column of T_i that occur in one row of T_j. Thus by argument given in Theorem 5.12, $\varepsilon_{ij} = 0$. This implies that the matrix A_I is an upper triangular matrix with 1's on the diagonal. So we have the following important result:

Corollary 5.15. The matrix $A_I = (\varepsilon_{ij})$ is invertible.

Basis for the Group Ring

In this section, we will use the results of the previous two sections to prove the important fact that all the $e_i S_{ij}$'s of all the standard tableaus of all the frames together form a basis for the group ring.

Theorem 5.16. All the $e_i^\lambda S_{ij}^\lambda$'s belonging to the standard tableaus of all the frames of a given degree n are linearly independent. λ refers to the corresponding frame.

Proof. Let k be the number of the frames of degree n. Then any linear dependence relation among the $e_i^\lambda S_{ij}^\lambda$ can be expressed as

$$\overline{A_1} + \ldots + \overline{A_r} + \ldots + \overline{A_k} = 0$$

Multiply the above equation on the left by $\overline{E_{ii}{}^r \; A_I{}^{-1}}$ and on the right by $\overline{A_I{}^{-1} \; E_{jj}{}^r}$, where A_I is (ε_{ij}) matrix for the frame F_r. By Theorem 5.3, the e_i's of the tableaus of the different frames annihilate each other. Therefore, we get:

$$\overline{E_{ii}{}^r \; A_I{}^{-1} \; A_I \; A_r \; A_I \; A_I{}^{-1} \; E_{jj}{}^r} \; = \; 0$$

or:

$$\overline{E_{ii}{}^r \; A_r \; E_{jj}{}^r} \; = \; 0$$

or:

$$\alpha_{ij}{}^r e_i{}^r S_{ij}{}^r \; = \; 0$$

Since $e_{ij}{}^r S_{ij}{}^r \neq 0$, we must have $\alpha_{ij}{}^r = 0$ for any choice of i, j, and r. Therefore, $e_i{}^r S_{ij}{}^r$ are linearly independent.

□

Thus for a fixed frame F_λ with f_λ standard tableaus, we have ${f_\lambda}^2$ linearly independent elements which by definition span the subring F^λ. So we have the following result:

Corollary 5.17. All the ${e_i}^\lambda {S_{ij}}^\lambda$'s of the frame F_λ form a basis for the subring F^λ, which then has dimension ${f_\lambda}^2$.

We have also proved the following important theorem:

Theorem 5.18. The sum of the subrings $F^1 + ... + F^k$ for a degree n with k frames is a direct sum. The dimension of this subring is $\sum_{j=1}^{k} {f_j}^2$.

In order to show that this direct sum is indeed the group ring O, we need to show that $\sum_{j=1}^{k} {f_j}^2$ equals the dimension of the group ring which is $n!$. But first we need to state a few definitions and lemmas.

Let F be a frame of type $(m_1, ..., m_r)$ with r rows and n boxes. Then we let $f = f(m_1, ..., m_r)$ be the number of the standard tableaus of F. We can create a new structure $F_{[i]}$ by removing the last box of the i–th row of F for each $i = 1, ..., r$. This new structure is a frame if and only if $m_i > m_{i+1}$. For the frame F,we can extend the definition of $f(m_1, ..., m_r)$ to the following:

$$f_{[i]} \;=\; f(m_1, ..., m_{i-1}, m_i - 1, m_{i+1}, ..., m_r)$$

$$= \begin{cases} 0 & \text{if} \quad m_i = m_{i+1} \\ \text{number of the standard tableaus of } F_{[i]} & \text{if} \quad m_i > m_{i+1} \end{cases} \tag{5.4}$$

Similarly, we can create a new structure $F^{[j]}$ by adding a new box to the end of the j-th row of F for $j = 1, ..., r+1$. This new structure is a frame if and only if $m_j < m_{j-1}$. Thus we can define:

$$\begin{aligned} f^{[j]} &= f(m_1, ..., m_{j-1}, m_j + 1, m_{j+1}, ..., m_r) \\ &= \begin{cases} 0 & \text{if} \quad m_j = m_{j-1} \\ \text{number of the standard tableaus of } F^{[j]} & \text{if} \quad m_j < m_{j-1} \end{cases} \end{aligned} \tag{5.5}$$

Lemma 5.19. $f(m_1, ..., m_r) = \sum_{i=1}^{r} f_{[i]}$.

Proof. On the right side, we have the number of the standard tableaus of the frame F which belongs to the group S_n. The largest number appearing in a box of a standard tableau is n, and it must occur in the last box of some row $i \leq r$ such that $m_i > m_{i+1}$. If we keep n fixed, then we get a frame which belongs to the group S_{n-1}, and we can create all the standard tableaus of this frame using the numbers $1, ..., n-1$. The number of the standard tableaus of this frame is given by $f_{[i]}$ which appears on the right side of the equality. So the left side is less than or equal to the right side. The remaining terms of the right side are all zero since they refer to the structures for which $m_i = m_{i+1}$. Therefore, the two sides are equal.

□

The definitions of $f(m_1, ..., m_r)$, $f_{[i]}$, and $f^{[j]}$ may be generalized to any structure F of the type $(m_1, ..., m_r)$. If F is a frame, then f, $f_{[i]}$, and $f^{[j]}$ are all well defined. If F is not a frame, then we define $f(m_1, ..., m_r), f_{[i]}$, and $f^{[j]}$ all to be zero, even if $f_{[i]}$ or $f^{[j]}$ is a frame. Then $(F^{[j]})_{[i]}$ refers to the structure created by removing the last box of the i–th row of $F^{[j]}$, and $(f^{[j]})_{[i]}$ denotes the number of the standard tableaus of the mentioned structure provided that both $F^{[j]}$ and $(F^{[j]})_{[i]}$ are frames, and it is zero otherwise. Similarly, we can define $(F_{[i]})^{[j]}$ and $(f_{[i]})^{[j]}$. Although $(F_{[i]})^{[j]}$ and $(F^{[j]})_{[i]}$ are the exact same structures, $(f_{[i]})^{[j]}$ and $(f^{[j]})_{[i]}$ may be different. For instance, for the frame of the type $(1, 1)$, $(f_{[1]})^{[1]} = 0$ while $(f^{[1]})_{[1]} = 1$. Throughout the remaining of this section, we assume that F is a structure of the type $(m_1, ..., m_r)$ with n boxes and r rows.

Lemma 5.20. Let F be a frame. Then the following results follow directly from Lemma 5.19 and the previous comments:

(1) $(f^{[r+1]})_{[r+1]} = f$

(2) $f^{[j]} = \sum_{i=1}^{r} (f^{[j]})_{[i]}$

(3) $f^{[r+1]} = \sum_{j=1}^{r+1} (f^{[r+1]})_{[i]}$

Lemma 5.21. Let F be a frame. If $F^{[j]}$ is not a frame and $i \neq j$, then $(f^{[j]})_{[i]} = (f_{[i]})^{[j]} = 0$.

Proof. Since F is a frame and $F^{[j]}$ is not, we must have $m_j = m_{j-1}$ for the frame F. Thus the j–th row of the structure $F^{[j]}$ is one box longer than its $(j-1)$–th row. The structure $(F^{[j]})_{[i]}$ is created by removing the last box of the i–th row of $F^{[j]}$. Since $i \neq j$, no box is removed from the j–th row of this structure. Therefore, the j–th row of the structure $(F^{[j]})_{[i]}$ or $(F_{[i]})^{[j]}$ is longer than its $(j-1)$–th row. Hence it is not a frame, and we have $(f^{[j]})_{[i]} = (f_{[i]})^{[j]} = 0$.

□

Lemma 5.22. Let F be a frame. If both $F^{[j]}$ and $(F^{[j]})_{[i]}$ are frames and $i \neq j$, then $(f^{[j]})_{[i]} = (f_{[i]})^{[j]}$.

Proof. Since both $(F^{[j]})_{[i]}$ and $(F_{[i]})^{[j]}$ lead to the same structure, we only need to show that $F_{[i]}$ is a frame. Assume on the contrary that $F_{[i]}$ is not a frame. Then we must have $m_i = m_{i+1}$ for the frame F. The frame $F^{[j]}$ is created by adding a box to the end of the j–th row of the frame F which is different from the i–th row. So the length of the i–th row of $F^{[j]}$ is less than or equal to the length of its $(i+1)$–th row. Now, we construct $(F^{[j]})_{[i]}$ by removing the last box of the i–th row of $F^{[j]}$. Therefore, the i-th row of the structure $(F^{[j]})_{[i]}$ is shorter than its $(i+1)$–th row. This means that $(F^{[j]})_{[i]}$ is not a frame which is a contradiction.

□

Let F be a frame of the type $(m_1, ..., m_r)$. Then k consecutive rows $r_i, ..., r_{i+k-1}$ of F form a block if:

(1): $m_i = m_{i+1} = ... = m_{i+k-1}$

(2): if $i > 1$, then $m_i < m_{i-1}$

(3): if $r_{i+k-1} < r$, then $m_{i+k-1} > m_{i+k}$

For instance, a tableau of type (4, 3, 3, 3, 2, 2) has three blocks. The first block includes the first row only; the second block includes the second to the forth row, and the third block includes the last two rows.

Lemma 5.23. Suppose the rows $r_i, ..., r_{i+k-1}$ form a block for the frame F. Then $\sum_{j=i}^{i+k-1}(f^{[j]})_{[j]} = \sum_{j=i}^{i+k-1}(f_{[j]})^{[j]} = f$.

Proof. We consider two cases: either $k = 1$ or $k > 1$.
Case 1: If $k = 1$, then $m_{i+1} < m_i < m_{i-1}$. Hence both $F^{[i]}$ and $F_{[i]}$ are frames. We also have $(F^{[i]})_{[i]} = (F_{[i]})^{[i]} = F$. These imply that $(f^{[i]})_{[i]} = (f_{[i]})^{[i]} = f$.
Case 2: Assume that $k > 1$. First, we show that $\sum_{j=i}^{j+k-1}(f^{[j]})_{[j]} = f$. Since $m_j = m_{j-1}$ for $j = i+1, ..., i+k-1$, none of the strucrtures $F^{[i+1]}, ..., F^{[i+k-1]}$ is a frame. So we have:

$$(f^{[i+1]})_{[i+1]} = ... = (f^{[i+k-1]})_{[i+k-1]} = 0$$

On the other hand, $f^{[i]}$ is a frame since either $m_i < m_{i-1}$ or r_i is the first row of the frame F. Hence:

$$\sum_{j=i}^{i+k-1}(f^{[j]})_{[j]} = (f^{[i]})_{[i]} = f$$

Now, we show that $\sum_{j=i}^{i+k-1}(f_{[j]})^{[j]} = f$. Since $m_j = m_{j+1}$ for $j = i, ..., i+k-2$, none of the structures $F_{[i]}, ..., F_{[i+k-2]}$ is a frame. The structure $F_{[i+k-1]}$, however, is a frame since r_{i+k-1} is either the last row of the frame or we have $m_{i+k-1} > m_{i+k}$. Therefore, we have:

$$\sum_{j=i}^{i+k-1} (f_{[j]})^{[j]} = (f_{[i+k-1]})^{[i+k-1]} = f$$

□

Lemma 5.24. If F is a frame with r rows, then $\sum_{i=1}^{r}(f^{[i]})_{[i]} = \sum_{i=1}^{r}(f_{[i]})^{[i]}$.

Proof. Applying Lemma 5.23, we get:

$$\begin{aligned} \sum_{i=1}^{r}(f^{[i]})_{[i]} &= \sum_{i=1}^{r}(f_{[i]})^{[i]} \\ &= b.f \end{aligned}$$

where b is the number of the blocks of F.

□

Lemma 5.25. If F is a frame with r rows, then $(n+1)f = \sum_{j=1}^{r+1} f^{[j]}$.

Proof. The proof is by induction. By definition of $f^{[j]}$, we hve $\sum_{j=1}^{r+1} f^{[j]} = \sum_{j=1}^{r+l} f^{[j]}$ for any $l > 1$. Clearly, the equality holds for $n = 1$. Now, we assume that it holds for any frame with $n-1$ boxes. In particular, we assume that it holds for any frame created by removing the last box of the i-th row of the frame F. Therefore, the induction hypothesis implies that:

$$nf_{[i]} = \sum_{j=1}^{r+1}(f_{[i]})^{[j]} \qquad \text{for} \qquad i = 1, ..., r$$

By Lemma 5.19, we have:

$$\sum_{j=1}^{r+1} f^{[j]} = \sum_{j=1}^{r+1} \sum_{i=1}^{r+1} (f^{[j]})_{[i]}$$

Hence:

$$\begin{aligned}
\sum_{j=1}^{r+1} f^{[j]} &= \sum_{j=1}^{r+1} \sum_{i=1}^{r} (f^{[j]})_{[i]} \quad + \quad (f^{[r+1]})_{[r+1]} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r+1} (f^{[j]})_{[i]} \quad + \quad (f^{[r+1]})_{[r+1]} \\
&= \sum_{i=1}^{r} \{ \sum_{j=1, i \neq j}^{r+1} (f^{[j]})_{[i]} \quad + \quad \sum_{j=1}^{r+1} (f^{[j]})_{[j]} \} \ + \ (f^{[r+1]})_{[r+1]}
\end{aligned}$$

It follows from Lemmas 5.21 and 5.22 that $(f^{[j]})_{[i]} = (f_{[i]})^{[j]}$ if $i \neq j$. Lemma 5.24 implies that $\sum_{j=1}^{r} (f^{[j]})_{[j]} = \sum_{j=1}^{r} (f_{[j]})^{[j]}$, and finally, by Lemma 5.20, we have $(f^{[r+1]})_{[r+1]} = f$. Therefore, the equality simplifies into the following:

$$\sum_{j=1}^{r+1} f^{[j]} = \sum_{i=1}^{r} \{ \sum_{j=1, i \neq j}^{r+1} (f_{[i]})^{[j]} \quad + \quad \sum_{j=1}^{r+1} (f_{[j]})^{[j]} \} \quad + \quad f$$

or:

$$\sum_{j=1}^{r+1} f^{[j]} = \sum_{i=1}^{r} \sum_{j=1}^{r+1} (f_{[i]})^{[j]} \quad + \quad f$$

Applying the induction hypothesis to the right side of the above equality, we get:

$$\sum_{j=1}^{r+1} f^{[j]} = n \sum_{i=1}^{r} f_{[i]} \quad + \quad f$$

By Lemma 5.19, $\sum_{i=1}^{r} f_{[i]} = f$. Therefore:

$$\begin{aligned}
\sum_{i=1}^{r+1} f^{[j]} &= nf + f \\
&= (n+1)f
\end{aligned}$$

□

Theorem 5.26. Let $F_1, ..., F_k$ be all the frames of degree n, then $\sum_{i=1}^{k} {f_i}^2 = n!$, where f_i is the number of the standard tableaus of the frame F_i.

Proof. The proof is by induction on the degree n. Clearly, the equality to be proved holds for $n = 1$. Assume that the equality holds for n. Then we have:

$$\sum_{i=1}^{k} {f_i}^2 = n!$$

Let $G_1, ..., G_m$ be all the frames of degree $n+1$; g_j be the number of the standard tableaus of G_j, and $(g_j)_{[i]}$ be the number of the standard tableaus of $(G_j)_{[i]}$ if $(G_j)_{[i]}$ is a frame and zero otherwise. Then we need to show that:

$$\sum_{j=1}^{m} {g_j}^2 = (n+1)!$$

Consider the number $S = \sum g_u f_v$, where the sum is taken over all possible pairs of numbers g_u and f_u for which the frame G_u can be obtained by adding a box to F_v. Clearly, each g_i and each f_i appears at least once in the summation. We simplify the number S in two different ways. In the first approach, we remove boxes from the G_i's and then apply Lemma 5.19. Let r_i denote the number of the rows of the frame G_i. Then we have:

$$\begin{aligned} S &= \sum g_u f_v \\ &= \sum_{i=1}^{m} \{ \, g_i \cdot \sum_{j=1}^{r_i} (g_i)_{[j]} \, \} \\ &= \sum_{i=1}^{m} {g_i}^2 \end{aligned}$$

In the second approach, we add boxes to the F_i's. Then we apply Lemma 5.25, and finally, we use the induction hypothesis. Let s_i denote the number of the rows of the frame F_i. Then we have:

$$\begin{aligned} S &= \sum g_u f_v \\ &= \sum_{i=1}^{k}\{\sum_{j=1}^{s_i+1}(f_i)^{[j]}\}.f_i \\ &= \sum_{i=1}^{k}\{(n+1)f_i\}.f_i \\ &= (n+1)\sum_{i=1}^{k}{f_i}^2 \\ &= (n+1)n! \\ &= (n+1)! \end{aligned}$$

So we have shown that $S = \sum_{i=1}^{m} {g_i}^2 = (n+1)!$, which concludes the induction step and proves the theorem.

□

We can conclude this section with the following theorem which is crucial to the approach we have ued in our procedure.

Theorem 5.27. All the $e_i S_{ij}$ of all the frames of degree n form a basis for the group ring. Furthermore, we have $O_{S_n} \simeq F^1 \oplus ... \oplus F^k$, where k is the number of the frames of degree n, and F^λ is the subring generated by the ${e_i}^\lambda {S_{ij}}^\lambda$'s.

Proof. By Theorem 5.16 all the $e_i S_{ij}$ of all the frames are linearly independent. The total number of all the $e_i S_{ij}$'s for a given frame is $\sum_{i=1}^{k} {f_i}^2$, which by Theorem 5.26 is $n!$. We have $n!$ linearly independent elements in a vector space O_{S_n} of dimension $n!$. So they form a basis for the group ring. We then know that $O_{S_n} = F^1 + F^2 + \ldots + F^k$. Since all the $e_i S_{ij}$ of all the frames are linearly independent, the sum is direct.

□

Independent Generating Set for the Multilinear Identities and Central Identities

In this section, we will show that the multilinear identities and central identities of degree n found by the procedure described in the second chapter form an independent generating set for all the multilinear identities and central identities of degree n. Since the argument for the multilinear central identities is very similar to that of the multilinear identities, we will state the proofs only for the multilinear identities. Also, in the rest of this section, an identity always means a degree n identity of the ϕ- algebra $M_3(\phi)$, where ϕ is a field of characteristic zero or greater than n. In chapter 4, we showed that every such identity is implied by a set of multilinear identities. We will now show that all the multilinear identities found by the procedure form an independent generating set. We will show that all the identities of degree $n < 9$ can be obtained from these identities. This shows that we have found all the identities of degree < 9. As mentioned before, a multilinear identity of degree n is an identity of the form $I(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\pi [x_1, \ldots, x_n]_\pi$, for some α_π's $\in \phi$. We may refer to this multilinear polynomial identity by the element $\sum_{\pi \in S_n} \alpha_\pi \pi$ of the group ring. Also in the rest of this section, we let $F_1, \ldots, F_k$ be all the frames of degree n and f_λ be the number of the standard tableaus of F_λ. Then an identity of the frame F_λ is of the form:

$$I(x_1, ..., x_n) = \sum_{i=1,j=1}^{f_\lambda, f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda \tag{5.6}$$

The procedure, on the other hand, computes all the identities of the form:

$$I(x_1, ..., x_n) = \sum_{i=1}^{f_\lambda} \alpha_{i1} e_i{}^\lambda S_{i1}{}^\lambda \tag{5.7}$$

The matrix representation of an identity of the form (5.7) is given by:

$$\begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \alpha_{f_\lambda 1} & 0 & \cdots & 0 \end{bmatrix} \tag{5.8}$$

We also know that if $I(x_1, ..., x_n)$ is an identity, then so is $I\pi(x_1, ..., x_n) = I(x_{\pi^{-1}1}, ..., x_{\pi^{-1}n})$ for any permutation $\pi \in S_n$. Furthermore, if $g = \sum_{\pi \in S_n} \alpha_\pi \pi$, then $Ig(x_1, ..., x_n) = \sum_{\pi \in S_n} \alpha_\pi I\pi(x_1, ..., x_n)$ is a linear combination of identities, and therefore, it is also an identity. In order to show that the procedure computes all of the identities of a given frame F_λ, we need to show that every identity of the form (5.6) can be obtained from all the identities of the form (5.7). Before we show this, we need to prove two lemmas. In the first lemma, we will show that if $I(x_1, ..., x_n)$ is an identity with the following matrix representation

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1f_\lambda} \\ \vdots & \ddots & \vdots \\ \alpha_{f_\lambda 1} & \cdots & \alpha_{f_\lambda f_\lambda} \end{bmatrix} ,$$

then for each $j = 1, \ldots, f_\lambda$, the group ring element with the following matrix representation is also an identity:

$$\begin{bmatrix} 0 & \cdots & 0 & \alpha_{1j} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{f_\lambda j} & 0 & \cdots & 0 \end{bmatrix}$$

In the second lemma, we will show that if $I_l(x_1, \ldots, x_n)$ is an identity of the frame F_λ with the following matrix representation:

$$\begin{array}{c} l\text{–th column} \\ \downarrow \\ \begin{bmatrix} 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{f_\lambda} & 0 & \cdots & 0 \end{bmatrix} \end{array},$$

then for each $m = 1, \ldots, f_\lambda$, the group ring element represented by the following matrix is also an identity:

$$\begin{array}{c} m\text{–th column} \\ \downarrow \\ \begin{bmatrix} 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{f_\lambda} & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

Lemma 5.28. If $I(x_1, ..., x_n) = \sum_{i=1,j=1}^{f_\lambda, f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda$ is an identity of the frame F_λ, then so is $I_j(x_1, ..., x_n) = \sum_{i=1}^{f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda$ for each $j = 1, ..., f_\lambda$.

Proof. Let $\overline{A} = \overline{[\alpha_{ij}]}$ be the matrix representation of $I(x_1, ..., x_n)$, and $g = \overline{A_I{}^{-1}\ E_{jj}}$, where A_I is the (ε_{ij}) matrix for the frame F_λ, and E_{jj} is the unit matrix with a 1 entry at the jj-th position and zeros elsewhere. Then $\overline{A} * g$ is also an identity. According to the multiplication rule given in section 2 of this chapter, we have:

$$\begin{aligned} \overline{A} * g &= \overline{A} * \overline{A_I{}^{-1} E_{jj}} \\ &= \overline{A\ A_I\ A_I{}^{-1}\ E_{jj}} \\ &= \overline{A\ E_{jj}} \\ &= \sum_{i=1}^{f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda \end{aligned}$$

Thus $\sum_{i=1}^{f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda$ ia also an identity.

□

Lemma 5.29. If $I_l(x_1, ..., x_n) = \sum_{i=1}^{f_\lambda} \alpha_{il} e_i{}^\lambda S_{il}{}^\lambda$ is an identity of the frame F_λ , then so is $I_m(x_1, ..., x_n) = \sum_{i=1}^{f_\lambda} \alpha_{il} e_i{}^\lambda S_{im}{}^\lambda$ for all $m = 1, ..., f_\lambda$.

Proof. Let A be the matrix with only one nonzero column such that $\overline{A} = I_l(x_1, ..., x_n)$ and $g = \overline{A_I{}^{-1} E_{lm}}$, where A_I is the (ε_{ij}) is the matrix for the F_λ frame, and E_{lm} is the unit matrix with a 1 entry at the lm-th position and zeros elsewhere. Then $Ig(x_1,...,x_n)$ or $\overline{A}*g$ is also an identity, and we have:

$$\begin{aligned}\overline{A}*g &= \overline{A} * \overline{{A_I}^{-1}E_{lm}}\\ &= \overline{A\,A_I\,{A_I}^{-1}\,E_{lm}}\\ &= \overline{A\,E_{lm}}\\ &= \sum_{i=1}^{f_\lambda} \alpha_{il}{e_i}^\lambda {S_{im}}^\lambda\end{aligned}$$

This means that the latter is an identity.

□

Theorem 5.30. If $I(x_1,...,x_n)$ is an identity of the frame F_λ , then for some g_m's $\in O$, we have $I(x_1,...,x_n) = \sum_{m=1}^{f_\lambda} L_m g_m(x_1,...,x_n)$, where each $L_m(x_1,...,x_n)$ is an identity of the form $\sum_{j=1}^{f_\lambda} \beta_{i1}{e_i}^\lambda {S_{i1}}^\lambda$ for some $\beta_{11},...,\beta_{\lambda_f 1} \in \phi$.

Proof. The strategy is to show that if $I(x_1,...,x_n) = \overline{[\alpha_{ij}]}$ is an identity of the frame F_λ, then each column of I itself is an identity. The location of these individual coulmns is immaterial. So we can represent each of them by putting the nonzero column first. In other words, $I(x_1,...,x_n)$ is equivalent to the identities represented by the following set of matrices

$$\left\{ \begin{array}{c} \begin{array}{ccccccc} & & & j\text{–th column} & & & \\ & & & \downarrow & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & \alpha_{1j} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{f_\lambda j} & 0 & \cdots & 0 \end{bmatrix} \end{array} : j = 1, \dots, f_\lambda \right\} ,$$

which in turn is equivalent to the identities represented by the following set of matrices

$$\left\{ \begin{array}{c} \begin{array}{cccc} \text{first column} & & & \\ \downarrow & & & \end{array} \\ \begin{bmatrix} \alpha_{1j} & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ \alpha_{\lambda_f j} & 0 & \cdots & 0 \end{bmatrix} \end{array} : j = 1, \dots, f_\lambda \right\} .$$

Now, we rigorously prove the theorem. We have:

$$\begin{aligned} I(x_1, \dots, x_n) &= \sum_{i=1,j=1}^{f_\lambda, f_\lambda} \alpha_{ij} e_i{}^\lambda S_{ij}{}^\lambda \\ &= I_1(x_1, \dots, x_n) + \dots + I_m(x_1, \dots, x_n) + \dots + I_{f_\lambda}(x_1, \dots, x_n) \quad , \end{aligned}$$

where $I_m(x_1, \dots, x_n) = \sum_{i=1}^{f_\lambda} \alpha_{im} e_i{}^\lambda S_{im}{}^\lambda$ for $m = 1, \dots, f_\lambda$. By Lemma 5.28, each $I_m(x_1, \dots, x_n)$ is an identity. Then by Lemma 5.29, $L_m(x_1, \dots, x_n) = \sum_{i=1}^{f_\lambda} \alpha_{im} e_i{}^\lambda S_{i1}{}^\lambda$ is an identity for each m. Now, let $g_m = \overline{A_I{}^{-1} E_{1m}}$ and $L_m(x_1, \dots, x_n) = \overline{\Gamma}$. Then for each $i = 1, \dots, f_\lambda$, we have:

$$\begin{aligned} L_m(x_1, \dots, x_n) \, * \, g_m &= \overline{\Gamma} \, * \, \overline{A_I{}^{-1} E_{1m}} \\ &= \overline{\Gamma E_{1m}} \end{aligned}$$

$$= \sum_{i=1}^{f_\lambda} \alpha_{im} e_i{}^\lambda S_{im}{}^\lambda$$
$$= I_m(x_1, ..., x_n)$$

Therefore, we have:

$$I(x_1, ..., x_n) = \sum_{m=1}^{f_\lambda} L_m(x_1, ..., x_n) * g_m \quad ,$$

where $L_m = \sum_{i=1}^{f_\lambda} \alpha_{im} e_i{}^\lambda S_{im}{}^\lambda$.

□

The above theorem implies that every identity of the frame F_λ is implied by the set of all the identities of the form $\sum_{i=1}^{f_\lambda} \alpha_{i1} e_i{}^\lambda S_{i1}{}^\lambda$. Thus the identities computed by the procedure imply all the identities of the given frame. Now, we need to show that every multilinear identity is implied by the set of all the identities of all of the frames.

Theorem 5.31. If $I(x_1, ..., x_n)$ is a multilinear identity of degree n, then $I(x_1, ..., x_n) = I_1(x_1, ..., x_n) + ... + I_k(x_1, ..., x_n)$, where $I_m(x_1, ..., x_n)$ is an identity of the frame F_m for each $m = 1, ..., k$.

Proof. First, notice that in this proof I_m refers to the whole m–th summand which is different from the notation used in theorem 5.30. By Theorem 5.27, $I(x_1, ..., x_n)$ can be expressed in the form

$$\sum_{i=1,j=1}^{f_1,f_1} \alpha_{ij}{}^1 e_i{}^1 S_{ij}{}^1 + ... + \sum_{i=1,j=1}^{f_m,f_m} \alpha_{ij}{}^m e_i{}^m S_{ij}{}^m + ... + \sum_{i=1,j=1}^{f_k,f_k} \alpha_{ij}{}^k e_i{}^k S_{ij}{}^k$$

for some coefficients $\alpha_{ij}{}^m \in \phi$; $m = 1, ..., k$. Let

$$I_m(x_1, ..., x_n) = \sum_{i=1,j=1}^{f_m,f_m} \alpha_{ij}{}^m e_i{}^m S_{ij}{}^m \qquad \text{for} \quad m = 1, ..., k$$

We need to show that each $I_m(x_1, ..., x_n)$ is an identity. Since $I(x_1, ..., x_n)$ is an identity, then so is $Ig(x_1, ..., x_n)$ for any $g \in O$. In particular, let $g = \overline{A_I}^{-1}$, where $A_I = (\varepsilon_{ij})$ for the frame F_m. Also let A_m be the matrix such that $\overline{A_m} = I_m(x_1, ..., x_n)$. Then the identity I is represented by $(\overline{A_1} + ... + \overline{A_m} + ... + \overline{A_k})$, and the identity Ig is represented by $(\overline{A_1} + ... + \overline{A_m} + ... + \overline{A_k}) * \overline{A_I^{-1}}$. Using the multiplication rule, we get:

$$\begin{aligned} Ig(x_1, ..., x_n) &= (\overline{A_1} + ... + \overline{A_m} + ... + \overline{A_k}) * \overline{A_I^{-1}} \\ &= \overline{A_m} * \overline{A_I^{-1}} \\ &= \overline{A_m\ A_I\ A_I^{-1}} \\ &= \overline{A_m} \end{aligned}$$

Therefore, $I_m(x_1, ..., x_n)$ is also an identity for each $m = 1, ..., k$.

□

So far we have shown that the procedure gives us all of the multilinear identities of degree n. Now, we need to show that these identities are independent under substitution, which in turn implies that they form an independent generating set for all of the multilinear identities of degree n. First, we will prove that the independent identities of the form 5.7 for a frame F_λ computed by the procedure form an independent generating set for all of the identities of that frame.

Lemma 5.32. A set of linearly independent identities of the form $\sum_{i=1}^{f_\lambda} \alpha_{i1} e_i{}^\lambda S_{i1}{}^\lambda$ for a given frame F_λ is also independent under substitution.

Proof. Let $I_1(x_1,...,x_n),...,I_\tau(x_1,...,x_n)$ be a set of linearly independent identities of the form $\sum_{i=1}^{\lambda}\alpha_{i1}e_i{}^{\lambda}S_{i1}{}^{\lambda}$ for a frame F_λ, where τ is the number of the independent identities in the set, and let the matrix representation of the identity $I_m(x_1,...,x_n)$ be given by:

$$A_m = \begin{bmatrix} \alpha_{11}{}^m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\lambda_f 1}{}^m & 0 & \cdots & 0 \end{bmatrix} \qquad \text{for} \quad m = 1,...,\tau$$

Assume on the contrary that the given identities are not independent under substitution. Then there exist nonzero elements $g_1,...,g_\tau \in O$ such that

$$I_1g_1(x_1,...,x_n) + ... + I_mg_m(x_1,...,x_n) + ... + I_\tau g_\tau(x_1,...,x_n) = 0$$

Since $O \cong F^1 + ... + F^k$, for $m = 1,...,\tau$ we have $g_m = (g_m)_1 + ... + (g_m)_k$ for some $(g_m)_i \in F^i$. Then for each $m = 1,...,\tau$, we have:

$$I_mg_m(x_1,...,x_n) = I_m(g_m)_1(x_1,...,x_n) + ... + I_m(g_m)_k(x_1,...,x_n)$$

Since the e_iS_{ij} of different frames annihilate one another, we get:

$$I_mg_m(x_1,...,x_n) = I_m(g_m)_\lambda(x_1,...,x_n) \qquad \text{for} \quad m = 1,...,\tau$$

Hence:

$$I_1(g_1)_\lambda(x_1,...,x_n) + ... + I_m(g_m)_\lambda(x_1,...,x_n) + ... + I_\tau(g_\tau)_\lambda(x_1,...,x_n) = 0$$

Let the matrix representation of $(g_m)_\lambda$ be given by $(G_m)_\lambda$ and let $A_I(G_m)_\lambda$ be given by the following matrix, where A_I is the (ε_{ij}) matrix for the frame F_λ.

$$A_I(G_m)_\lambda = \begin{bmatrix} {\beta_{11}}^m & \cdots & {\beta_{1\lambda_f}}^m \\ \vdots & \ddots & \vdots \\ {\beta_{\lambda_f 1}}^m & \cdots & {\beta_{\lambda_f\lambda_f}}^m \end{bmatrix} \qquad \text{for} \quad m = 1, \ldots, \tau$$

Then:

$$\begin{aligned} I_m g_m(x_1, \ldots, x_n) &= I_m(x_1, \ldots, x_n) \ * \ (g_m)_\lambda \\ &= \overline{A_m} \ * \ (g_m)_\lambda \\ &= \overline{A_m \ A_I \ (G_m)_\lambda} \end{aligned}$$

Then the matrix representation of $I_m g_m(x_1, \ldots, x_n)$ for each $i = 1, \ldots, \tau$ is given by:

$$\begin{bmatrix} {\beta_{11}}^m {\alpha_{11}}^m & \cdots & {\beta_{1\lambda_f}}^m {\alpha_{11}}^m \\ \vdots & \ddots & \vdots \\ {\beta_{11}}^m {\alpha_{\lambda_f 1}}^m & \cdots & {\beta_{1\lambda_f}}^m {\alpha_{\lambda_f 1}}^m \end{bmatrix}$$

We have assumed that $I_m g_m(x_1, \ldots, x_n) \neq 0$ for some m. Let l be a nonzero column for the matrix representation of $I_m g_m(x_1, \ldots, x_n)$. Then ${\beta_{1l}}^m \neq 0$ since every entry of the l-th column of the matrix of $I_m g_m(x_1, \ldots, x_n)$ is a multiple of ${\beta_{1l}}^m$. If we multiply both sides of $I_1 g_1(x_1, \ldots, x_n) + \ldots + I_\tau g_\tau(x_1, \ldots, x_n) = 0$ on the right by $\overline{{A_I}^{-1} E_{l1}}$, we get:

$$\{\sum_{m=1}^{\tau} I_m g_m(x_1, \ldots, x_n)\} \ * \ \overline{{A_I}^{-1} E_{l1}} = 0$$

or:

$$\sum_{m=1}^{\tau} \overline{A_m A_I (G_m)_\lambda} * \overline{{A_I}^{-1} E_{l1}} = 0$$

or:

$$\sum_{m=1}^{r} \overline{A_m A_I (G_m)_\lambda E_{l1}} = 0$$

But:

$$A_m A_I (G_m) E_{l1} = \begin{bmatrix} \alpha_{11}{}^m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\lambda_f 1}{}^m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \beta_{1l}{}^m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{\lambda_f l}{}^m & 0 & \cdots & 0 \end{bmatrix}$$

$$= \beta_{1l}{}^m A_m$$

Therefore, we have

$$\beta_{1l}{}^1 I_1(x_1, ..., x_n) + ... + \beta_{1l}{}^m I_m(x_1, ..., x_n) + ... + \beta_{1l}{}^r I_r(x_1, ..., x_n) = 0$$

which implies that $I_1(x_1, ..., x_n), ..., I_r(x_1, ..., x_n)$ are linearly dependent, and that is a contradiction.

□

Recall that all the identities chosen for a fixed frame using the procedure are linearly independent. In fact, the number of these identities equals the nullity of the matrix given by the matrix of 2.28. Thus according to the above theorem all of the identities of a frame F_λ found by the procedure form an independent generating set for all of the identities of that frame. Now, we need to show that all of the identities of all of the frames given by the procedure form an independent generating set for all of the multilinear identities of degree n. This is accomplished by the following theorem which concludes this chapter.

Theorem 5.33. Let each H_λ be a set of linearly independent identities of the form $\sum_{i=1}^{\lambda'} \alpha_{i1}{}^\lambda e_i{}^\lambda S_{i1}{}^\lambda$ for each $\lambda = 1, ..., k$, where k is the number of the frames of degree n. Then the set $\cup_{\lambda=1}^k H_\lambda$ is independent under substitution.

Proof. Let all the identities of the set H_λ be given by $I_1{}^\lambda, ..., I_{\tau_\lambda}{}^\lambda$ where τ_λ denotes the cardinality of the set H_λ for each $\lambda = 1, ..., k$. Assume on the contrary that the elements of the set $\cup_{\lambda=1}^k H_\lambda$ are not linearly independent under substitution. Then there exist nonzero elements $g_m{}^\lambda$ of the group ring such that:

$$\sum_{\lambda=1}^{k} \sum_{m=1}^{\tau_\lambda} I_m{}^\lambda g_m{}^\lambda(x_1, ..., x_n) = 0$$

The linear independence of all of the $e_i S_{ij}$'s of all of the frames implies that:

$$\sum_{m=1}^{\tau_\lambda} I_m{}^\lambda g_m{}^\lambda(x_1, ..., x_n) = 0 \qquad \text{for} \quad \lambda = 1, ..., k$$

In other words, the linearly independent identities of each set H_λ are also independent under substitution which contradicts Lemma 5.32.

□

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